習題演習
傅利葉級數

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■ 週期爲 2π 及任意週期之 Fourier 級數

1. 若 $f(x+2\pi)=f(x)$，且 $f(x)=\begin{cases}-1, & -\pi \le x \le 0\\ 1, & 0<x<\pi\end{cases}$，求其 Fourier 級數。

【91 交大土木 15%】

【參考解答】 $f(x)=\sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\sin nx=\frac{4}{\pi}[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+....]$

2. (1) Find the Fourier series of the given function with period of 2π.

$$f(x)=\begin{cases}0, & if \quad -\pi<x<0\\ x^2, & if \quad 0<x<\pi\end{cases}$$

(2) Is Fourier series differentiable term by term? 【91 暨南電機 10%】

【參考解答】

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2(-1)^n}{n^2}$$

$$b_n=\frac{\pi}{n}(-1)^{n+1}-\frac{2}{n^3\pi}[1-(-1)^n]$$

$$f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}[\frac{\pi}{n}(-1)^{n+1}-\frac{2}{n^3\pi}(1-(-1)^n)]\sin nx$$

3. Find the Fourier series of the following periodic function $f(t)$.

$$f(t)=\begin{cases}1+t^2, & o<t<1\\ 3-t, & 1<t<2\end{cases}$$ ， $f(t+2)=f(t)$ 【91 交大機械 10%】

【參考解答】 $f(t)=\frac{17}{6}+\sum_{n=1,3,5}^{\infty}\frac{-4}{n^2\pi^2}\cos n\pi t+\sum_{n=1,3,5}^{\infty}\frac{-4}{n^3\pi^3}\sin n\pi t$

4. Consider the following 2π-period square wave function.

$f(x)=\begin{cases}1, & 0<x<\pi \\ 0, & -\pi<x<0\end{cases}$, $f(x)=0.5$ at $x=0$ and $x=\pm\pi$.

Let $s(x)=\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$ denote the Fourier series of $f(x)$ and

$s_N(x)=\sum_{n=0}^{N} a_n \cos nx + b_n \sin nx$ be its N^{th} - partial sum.

(1) Find a_n and b_n.

(2) Does $s_N \to f(x)$ as $N\to\infty$ at a fixed x, $-\pi<x<\pi$?

(3) Since $s(x)=f(x)$ and $\dfrac{df}{dx}=0$ for $0<x<\pi$, it is obvious that

$\dfrac{d}{dx}(s(x))=\sum_{n=0}^{\infty}(-na_n \sin nx + nb_n \cos nx)=0$ for $0<x<\pi$. Is the statement correct? You need to explain your answer briefly. 【89 台大機械 13%】

【參考解答】

(1) $f(x)=0.5+\sum_{n=1}^{\infty}\dfrac{1-(-1)}{n\pi}\sin nx$, $a_0=0.5$, $a_n=0$, $n\neq 0$, $b_n=\dfrac{1-(-1)^n}{n\pi}$.

(2) for $-\pi<x<\pi$，$s(x)\to f(x)$ for every point.

(3) $\dfrac{ds(x)}{dx}=\sum_{n=0}^{\infty}-na_n \sin nx + nb_n \cos nx = 0$, the statement is incorrect.

5. Let g be a periodic function defined by $g(t)=t^2$ for $0<t<3$ and $g(t+3)=g(t)$ for all t.

(1) Draw the graph of g for $-6<t<6$.

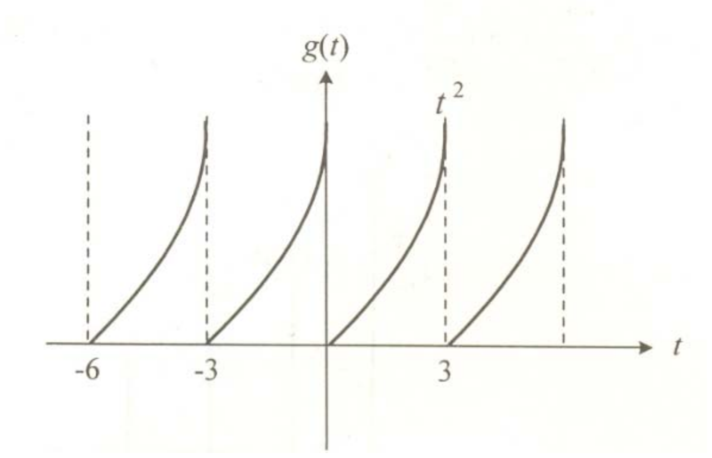
(2) Compute the Fourier series of g.

(3) Draw the amplitude spectrum of g for the three lowest-frequency components.

【91 台科電機 20%】

【參考解答】

(1)



(2) $g(t)=3+\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi t}{3}+\sum_{n=1}^{\infty}\frac{-9}{n\pi}sin\frac{2n\pi t}{3}$

(3) $g(t)=3+\sum_{n=1}^{\infty}c_n\cos(n\omega_0 t+\theta)$

6. Find the phase angle form of the Fourier series of $f(x)=\begin{cases}\cos\pi x, & 0\le x\le 1\\ f(x+1), & \forall x\in R\end{cases}$.

【91 北科冷凍 14%】

【參考解答】$C_n=\dfrac{8n}{4n^2-1}$ ，$\omega_0=2\pi$ ，$c_0=0$

■ 奇函數與偶函數之 Fourier Series

1. 求 $f(x)=|x|$ 於 $(-\pi,\pi)$ 內之 Fourier series，並求

(1) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+......$　(2) $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+......$

【91 中山材料 20%】【91 屏科機械 30%】

【參考解答】

(1) 取 $x=0$ ，得 $0=\frac{\pi}{2}-\frac{4}{\pi}[1+\frac{1}{3^2}+\frac{1}{5^2}+....]$

(2) 將 fourier consine series 兩邊同乘 x，並積分，得 $\frac{\pi^2}{96}=1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+....$

2. $f(x)=x^2, 0\le x\le 2\pi$，且 $f(x+2\pi)=f(x)$ 求 $f(x)$ 之 Fourier series，並求 $\sum_{n=1}^{\infty}\frac{1}{n^2}$。

【89 成大工程科學 15%】

【參考解答】$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

3. $f(x)=x^2, -\pi\le x\le\pi$ ，求：(1) Fourier series　(2) $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$ 。【91 交大機械 15%】

【參考解答】

取 $x=0$ 得 $-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+.....=-\frac{\pi^2}{12}$ ....(a)

$x=\pi$ 得 $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....=\frac{\pi^2}{6}$ ....(b)

由(b)-(a)得 $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....=\frac{\pi^2}{8}$

4. (1)Find the Fourier series of $f(t)$, $-\pi<t<\pi$.

(2)Show $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\frac{\pi}{4}$.

(3)With the series derived in part (1), show that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$. 【90 中山材料 10%】

【參考解答】

(1) $f(t)=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nt$

(2) at $t=\frac{\pi}{2}$, series converges to $\frac{\pi}{2}$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+......$$

(3) $\int_0^{\pi}t^2dt=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\int_0^{\pi}t\bullet\sin nt dt$, $\frac{\pi}{3}=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\bullet\frac{\pi}{n}(-1)^{n+1}$, $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$

5. (1) Find the Fourier series of the following periodic function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \sin x, & 0\le x\le\pi\end{cases}$$

(2) Using the derived result calculate

<1> $\frac{1}{1x3}+\frac{1}{3x5}+\frac{1}{5x7}+.....$ <2> $\frac{1}{1x3}-\frac{1}{3x5}+\frac{1}{5x7}-.....$

【91 成大機械 20%】【90 中山物理 20%】

【參考解答】

(1) $f(x)=\frac{1}{\pi}-\frac{2}{\pi}[\frac{1}{3}\cos 2x+\frac{1}{4^2-1}\cos 4x+\frac{1}{6^2-1}\cos 6x+...]+\frac{1}{2}\sin x$

(2) <1> $\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+......=-\frac{1}{2}$

<2> $\frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}-......=\frac{1}{4}(\pi-2)$

6. Represent the known function $y=|x-3|$ for $2<x<4$ by

(1) a Fourier series expansion.

(2) a Fourier sine series expansion and

(3) a Legendre polynomial expansion, respectively.

(4) Give a set of criteria and thereby judge which of the above three expansions is the best and which is better. 【88 台大土木 20%】

【參考解答】

(1) $y=\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1,3,5}^{\infty}\cos n\pi x$

(2) $y=\sum_{n=1,3,5}^{\infty}(\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{4}{n\pi})\sin\frac{n\pi x}{2}$

(3) 由於無法求出C_n之通式，無法比較收斂速度快慢。

7. (1) Find a Fourier series of period 6 which in interval (1,7) represents a function $f(x)$ taking on the constant value +1 when $1<x<4$ and constant value -1 when $4<x<7$.

(2) Reducing the above Fourier series to the following form:

$f(x)=A\sum_{nood}B\sin\frac{n\pi(x-1)}{3}$, what are the values of A and B? 【89 成大電機 12%】

【參考解答】

(1) 取 $f(t+1)$ 之 Fourier sine series 展開

$$f(x)=\sum_{n=1,3,5}^{\infty}\frac{-4}{n\pi}\sin\frac{n\pi}{3}\cos\frac{n\pi x}{3}+\sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\cos\frac{n\pi}{3}\sin\frac{n\pi x}{3}$$

(2) $A=\frac{4}{\pi}$，$b=\frac{1}{n}$。

8. 如果一函數$f(x)$在$0\le x\le 2$區間內之定義爲：$f(x)=\begin{cases}3, & for\ \ 0\le x\le 1\\ 1, & for\ \ 1<x\le 2\end{cases}$

(1) 將此函數在$0\le x\le 2$區間內以一週期爲 4 之 Fourier cosine series 來表示

(2) 利用(1)之結果推導一個級數和公式

(3) 將此函數在$0\le x\le 2$區間內以一個週期爲 2 之 Fourier series 來表示 【89 雲科營建 25%】

【參考解答】

(1) $f(x)=2+\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$

(2) $1=\frac{4}{\pi}[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}.....]$，$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}.....$

(3) $f(x)=2+g(x)=2+\sum_{n=1,3,5}^{\infty}\frac{4}{\pi}\sin n\pi x$

9. Find the Fourier series expansion for $f(t)$ and $|f(t)|$ with $f(t)=A\sin(\omega t+\phi)$, where A, ω and ϕ are all positive constants.【91 清大電機 10%】

【參考解答】

(1) $f(t)=A\sin(\omega t+\phi)=Ap[\sin\omega t\cos\omega\phi+\cos\omega t\bullet\sin\omega\phi]$

(2)

$|f(t)|=\frac{2A}{\pi}+\sum_{n=1}^{\infty}\frac{2A\cos n\phi}{\pi(1-n^2)}[1-(-1)^{n+1}]\bullet\cos\omega t+\sum_{n=1}^{\infty}\frac{2A}{\pi(n^2-1)}[1-(-1)^{n+1}]\bullet\sin n\phi\bullet\sin n\omega t$

10. Find Fourier series of $f(x)=\begin{cases}x, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$, $f(x+2\pi)=f(x)$.

【90 海洋船研通訊組 20%】

【參考解答】

$f(x)$ 爲週期是 2 的奇函數，取 Fourier sine series，$\therefore f(x)=\sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx$

11. (1) If $f(x)=\begin{cases}x-4, & 6\le x\le 9\\ x-10, & 9<x<12\end{cases}$, $f(x)=f(x+6)$, find Fourier series of $f(x)$.

(2) If $g(x)=\begin{cases}x-8, & 8\le x\le 11\\ x-14, & 11<x<14\end{cases}$, $g(x)=g(x+6)$, find series expansion of $g(x)$ in terms of an expression similar to Fourier series expansion.【台大土木 25%】

【參考解答】

(1) $f(x)=h(x)+2=2\sum_{n=1}^{\infty}\frac{6}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{3}$

(2) $g(x)=\sum_{n=1}^{\infty}\frac{6}{n\pi}(-1)^{n+1}\sin\frac{n\pi t}{3}=\sum_{n=1}^{\infty}\frac{6}{n\pi}(-1)^{n+1}\sin\frac{n\pi(x-2)}{3}$

■ 半幅展開

1. Expand $f(x)=\begin{cases}\frac{2h}{\ell}x, & 0\le x\le\frac{\ell}{2}\\ 2h-\frac{2h}{\ell}x, & \frac{\ell}{2}\le x\le\ell\end{cases}$ in a Fourier cosine series.

【91 中央電機 10%】

【參考解答】 $f(x)=\frac{h}{2}-\frac{4h}{\pi^2}[\cos\frac{2\pi x}{\ell}+\frac{1}{3^2}\cos\frac{6\pi x}{\ell}+\frac{1}{5}\cos\frac{10\pi x}{\ell}+....]$

2. Riemann zeta functions are defined as $\zeta(z)=\sum_{k=1}^{\infty}k^{-z}$, $\mathrm{Re}(z)>1$. By using half-range Fourier cosine series of $f(x)=x^2$, $0<x\le\pi$. Calculate $\zeta(2)$. Then integrate twice to calculate $\zeta(4)$. 【91 交大機械 15%】【91 清大物理 10%】

【參考解答】 $I=\frac{\pi^4}{90}=\zeta(4)$

3. Find the half-range cosine expansion and the half-range sine expansion of the function $f(t)=t^2$, $0\le t\le 1$. Which has the problem with uniform convergence (explain)? 【89 交大機械 17%】

【參考解答】

(1) $f(t)=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}\cos n\pi t$ ， $f(t)$連續，uniform convergence。

(2) $f(t)=\sum_{n=1}^{\infty}[\frac{2(-1)^{n+1}}{n\pi}-\frac{4[1-(-1)^n]}{n^3\pi^3}]\sin n\pi t$，sine series 無均勻收斂，即有 Gibbs 現象發生。

4. Let $f(x)=x$ for $0<x<1$.

(1) Expand $f(x)$ in Fourier cosine series for period 2.

(2) Expand $f(x)$ in Fourier sine series for period 2.

(3) Explain the relation of the solutions obtained form (1) and (2).

【91 成大機械 17%】

【參考解答】

(1) $f(x)=\frac{1}{2}+\sum_{n=1,3,5}^{\infty}\frac{-4}{n^2\pi^2}\cos n\pi x$

(2) $f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x$

(3) There is no relation of solutions obtained from (1) and (2)，勉強找些關係，只能說 Fourier cosine series 無 Gibbs 現象，收斂速度快，Fourier sine series 有 Gibbs 現象，收斂速度慢。

5. Given $f(x)=\begin{Bmatrix} \frac{2}{L}x+\frac{1}{2}, & when\ \ 0<x<\frac{L}{2} \\ \frac{4}{L}x+4, & when\ \ \frac{L}{2}<x<L \end{Bmatrix}$

(1) Find the half range Fourier expansion with odd periodic continuation of their function.

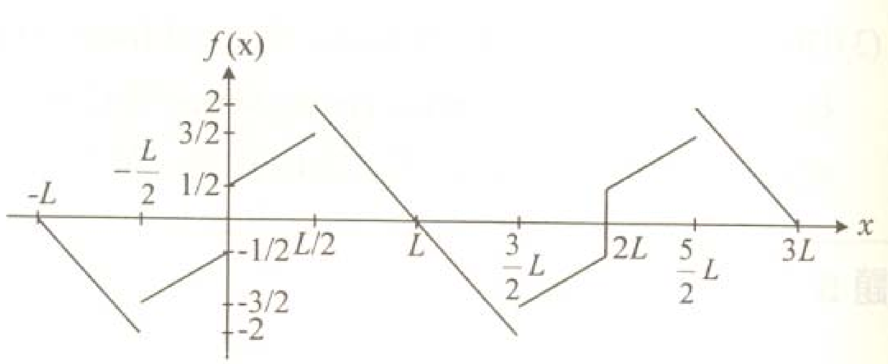
(2) Draw figure of your obtained series, including several cycles.

(3) Does your obtained series really represent the given function at every point between 0 and L inclusively, give comments (試討論之). 【91 中央光電 14%】

【參考解答】

(1) $f(x)=\sum_{n=1}^{\infty}[\frac{1}{n\pi}\cos\frac{n\pi}{2}+\frac{1}{n\pi}+\frac{12}{n^2\pi^2}\sin\frac{n\pi}{2}]\bullet\sin\frac{n\pi x}{L}$

(2)



(3) 在 $f(x)$連續之處，Fourier series 收斂到函數值，在不連續之點，series 收斂到平均值，在 $x=0$ 級數收斂到 0，在 $x=L/2$ 級數收斂到 $\frac{1}{2}(2+\frac{3}{2})$，$0\le x\le L$ 其餘處，級數收斂到 $f(x)$。

6. (1) Determine the coefficient in representation $f(x)=\sum_{n=1}^{\infty}A_n\sin nx,\ \ 0<x<\pi$,

$f(x) = 1.1.$

(2) Evaluate the value of the following series $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. 【87 中央太空 20%】

【參考解答】

(1) $f(x) = \sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\sin nx$

(2) $\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$

7. For a differentiable function y(x) defined on $0 \le x \le 1$, what values do the term-by-term differentiation at x = 0 of the

   (1) Fourier series
   (2) Fourier cosine series
   (3) Fourier sine series
   (4) Complex Fourier series of y(x) converge to x=0 respectively?【90 台大土木 16%】

【參考解答】

(1) Fourier series 逐項微分，當 y(0) = y(1)，x = 0 時，收斂到$\frac{1}{2}[y'(0)+y'(1)]$

$y(0) \ne y(1)$；當 x = 0，收斂到∞或$-\infty$，逐項微分有脈衝波出現。

(2) Fourier cosine series 逐項微分在 x = 0，收斂到 0

(3) Fourier sine series 逐項微分

當 y(0)=0，在 x=0，收斂到 $y'(0)$，當 $y(0) \ne 0$，在 x = 0，收斂到∞或$-\infty$，有脈衝波出現。

(4) complex Fourier series 與 Fourier series 相同。

8. $f(x) = \begin{Bmatrix} 1, & 0 \le x < 1 \\ 100, & x = 1 \\ 2, & 1 < x \le 2 \end{Bmatrix}$, suppose that the Fourier sine and Fourier cosine series of f(x) converge respectively to s(x) and g(x) on interval $0 \le x < 2$, without find the series, find s(x) and g(x). 【90 海洋機械 14%】

【參考解答】s(x)為週期 4 之奇函數，g(x)為週期 4 之偶函數。

$$s(x)=\begin{cases}0, & x=0\\ 1, & 0<x<1\\ \frac{3}{2}, & x=1\\ 2, & 1<x<2\\ 0, & x=2\end{cases},\quad g(x)=\begin{cases}1, & 0\le x<1\\ \frac{3}{2}, & x=1\\ 2, & 1<x\le 2\end{cases}$$

9. Find the Fourier series of the following function $f(x)=\begin{cases}x, & 0\le x<1\\ 0, & 1<x<2\end{cases}$.

【91成大造船10%】

【參考解答】$a_0=\frac{1}{2}\int_0^2 f(x)dx$，$a_n=\int_0^2 f(x)\cos n\pi x dx$，$b_n=\int_0^2 f(x)\sin n\pi x dx$

■ 複係數之 Fourier Series

1. Find the complex Fourier series of $f(x)=e^x$ if $-\pi<x<\pi$, $f(x+2\pi)=f(x)$, and obtain from it usual Fourier series.

【91交大機械25%】

【參考解答】$f(x)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\pi(1+n^2)}\sinh\pi\bullet(1-in)e^{inx}$ 爲 complex Fourier series

$f(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\bullet(-1)^n}{\pi(1+n^2)}\sinh\pi\bullet\sin nx$ 爲 usual Fourier series

2. Given that $f(x)=\begin{cases}0, & -\frac{1}{2}<x<-\frac{1}{4}\\ 1, & -\frac{1}{4}<x<\frac{1}{4}\\ 0, & \frac{1}{4}<x<\frac{1}{2}\end{cases}$ and $f(x)=f(x+1)$, find the complex Fourier series of $f(x)$ and plot points $(n,|c_n|)$ for $n=0,\pm1,\pm2,\ldots\ldots$.

【91元智電機控制組20%】

【參考解答】$c_n=\frac{1}{n\pi}\sin\frac{1}{2}n\pi, n\neq 0$, $f(x)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ n\neq -1}}^{\infty}\frac{1}{n\pi}\sin\frac{1}{2}n\pi\bullet e^{i2n\pi x}$

$$c_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \int_{-\frac{1}{4}}^{\frac{1}{4}} 1 \bullet dx = \frac{1}{2}$$

3. A function f(x) is defined in the range $[-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} 0 & for \ -\pi < x \le 0 \\ 1 & for \ 0 < x \le \pi \end{cases}$$

Expand f(x) into a complex Fourier series. 【91 清大電機 10%】

【參考解答】 $f(x) = \frac{1}{2} + \frac{2}{\pi}[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...]$ 爲 real Fourier series

4. Determine the complex exponential Fourier series coefficient c_n for the periodic function $f(t) = e^t$，$0 \le t < 1$，and which has the period T=1, and plot the complex Fourier spectrum $|c_n|$ versus $n\omega_0$. 【90 台科控制 5%】

【參考解答】 $|c_n| = (e-1) \bullet \frac{1}{\sqrt{1+4n^2\pi^2}}$ ， $n\omega_0 = 2n\pi$

■ Fourier 積分與 Fourier 轉換

1. (1) 函數 $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ 之 Fourier 積分式爲何？

(2) 由(1)推求 $\int_0^{\infty} \frac{\sin x \cos x}{x} dx$ =? 【91 嘉義土木 15%】

【參考解答】

(1) $f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \omega}{\omega}$ 爲所求

(2) $\frac{1}{2} = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \omega}{\omega} \bullet \cos \omega d\omega$ ， $\int_0^{\infty} \frac{\sin \omega \bullet \cos \omega}{\omega} d\omega = \frac{\pi}{4}$

2. (1) Find the Fourier integral representation of the following function.

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

(2) Use the result of (1) to show that $\int_0^\infty \frac{\sin 2x}{x} dx = \frac{\pi}{2}$. 【91 嘉義機電 30%】

【參考解答】

(1) $f(x) = \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}[\cos\omega x \sin 2\omega + \sin\omega x - \sin\omega x \cos 2\omega] d\omega$

(2) at x = 0 Fourier 積分值爲 $\frac{1}{2}$

$\frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\sin 2\omega d\omega$ ， $\therefore \int_0^\infty \frac{1}{x}\sin 2x dx = \frac{\pi}{2}$

3. Let x(t) be a rectangular pluse defined by x(t), $|t| < \frac{1}{2}$ and x(t)=0, otherwise. The corresponding Fourier transform is denoted as

$X(j\omega)$, $i.e.,$ $x(j\omega) = \int_0^\infty x(t)\exp(-j\omega t) dt$

(1) $X(j\omega)$ (2) $\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega$

(3) $\int_{-\infty}^{\infty} \frac{\sin\omega \cos(2\omega)}{\omega} d\omega$ (4) $\int_{-\infty}^{\infty} [x(j\omega)]^2 d\omega$ 【91 中山電機 20%】

【參考解答】

(1) $X(j\omega) = \frac{2}{\omega}\sin\frac{\omega}{2}$

(2) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\omega}\sin\frac{\omega}{2} e^{j\omega t} d\omega$

(3) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\omega}\sin\frac{\omega}{2}\cos x\omega t d\omega$

(4) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{i\omega t} d\omega$

4. Find the Fourier integral, k>0.

(1) $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$

(2) $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = -f(x)$ 【91 中山材料 20%】

【參考解答】

(1) $f(x) = \frac{2}{\pi}\int_0^\infty \frac{k}{\omega^2 + k^2}\cos\omega x dx$ 爲所求

(2) $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{k}{\omega^2+k^2}\sin\omega x dx$ 爲所求

5. Given the function shown as follows: $f(x)=\begin{cases}1+x, & -1<x\le 0\\ -(x-1), & 0<x\le 1\\ 0, & otherwies\end{cases}$

(1) Calculate the Fourier integral representation of the above function.

(2) Find $\int_0^{\infty}\frac{(1-\cos\omega)^2}{\omega^2}d\omega$.

(3) Compute $\int_0^{\infty}\frac{(1-\cos\omega)^2}{\omega^4}d\omega$. 【90 成大電機 15%】

【參考解答】

(1) $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{(1-\cos\omega)}{\omega^2}\bullet\cos\omega x\bullet d\omega$

(2) $\frac{\pi}{2}=\int_0^{\infty}\frac{1-\cos\omega}{\omega^2}d\omega$

(3) $\frac{\pi}{6}=\int_0^{\infty}\frac{(1-\cos\omega)^2}{\omega^4}d\omega$

6. 若 $f(x)=\begin{cases}1, & for\,|x|\le\frac{a}{2},\ a>0\\ 0, & elsewhere\end{cases}$

(1) 求 $f(x)$ 的 Fourier Transform *F(u)*, $F(u)=\int_{-\infty}^{\infty}f(x)\exp(-2\pi iux)dx.$

(2) Sketch $I(u)=|F(u)|^2$ versus frequency u. 【91 淡江物理 20%】

【參考解答】

(1) $F(u)=2\int_0^{\frac{a}{2}}\cos 2\pi ux dx=\frac{1}{\pi u}\sin a\pi u$

(2) $I(u)=|F(u)|^2=\frac{1}{\pi^2}\bullet\frac{1}{u^2}\sin^2 a\pi u=\frac{1}{2\pi^2}\frac{1}{u^2}[1-\cos 2\pi au]$，$I(-u)=I(u)$

7. 利用傅立葉積分證明 $\int_0^{\infty}\frac{\cos x\omega}{1+\omega^2}d\omega=\frac{\pi}{2}e^{-x}$，$x>0$。 【91 師大機電 15%】

【參考解答】 $\frac{\pi}{2}e^{-x}=\int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega$ 得證

8. Use Fourier integral to demonstrate the following results and show the details of your work.

$$\int_0^{\infty}\frac{\cos x\omega+\omega\sin x\omega d\omega}{1+\omega^2}=\begin{cases}0, & x<0\\ \frac{\pi}{2}, & x=0\\ \pi e^{-x}, & x>0\end{cases}$$

【91 彰師機械 15%】【90 中興化工 15%】

【參考解答】

依據 Dirichlet theorem 得知 $\int_0^{\infty}\frac{\cos x\omega+\omega\sin x\omega d\omega}{1+\omega^2}=\begin{cases}0, & x<0\\ \frac{\pi}{2}, & x=0\\ \pi e^{-x}, & x>0\end{cases}$

9. Prove $\int_0^{\infty}\frac{\omega^3\sin x\omega}{4+\omega^4}d\omega=\frac{\pi}{2}e^{-x}\cos x$ if $x>0$. 【91 中興化工 10%】

【參考解答】

依據 Dirichlet 定理得知 $\int_0^{\infty}\frac{\omega^3\sin x\omega}{4+\omega^4}d\omega=\frac{\pi}{2}e^{-x}\cos x$ ， $x>0$ ，得證。

10. 已知 $F\left\{e^{-at^2}\right\}=\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$ ， $F\left\{\frac{1}{a^2+t^2}\right\}=\frac{\pi}{a}e^{-a|\omega|}$ ，其中 a=常數，試求 $(64t^2-8)e^{-4t^2}$ 之傅立葉轉換。 【88 台科營建 17%】

【參考解答】 $F[(64t^2-8)e^{-4t^2}]=-\frac{1}{2}\sqrt{\pi}\omega^2\bullet e^{-\frac{\omega^2}{16}}$

11. Find the Fourier transform of the following function $f(x)$.

$$f(x)=\begin{cases}e^{x}, & x<0\\ e^{-x}, & x>0\end{cases}$$

【91 暨南電機 10%】

【參考解答】 $f(-x)=f(x)$ ， $F[f(x)]=2\int_0^\infty e^{-x}\cos\omega x dx=\dfrac{2}{\omega^2+1}$

12. Obtain the integration and the Fourier transform of a Gaussian function as expressed below.

(1) $\int_{-\infty}^{\infty}\exp[-ax^2]dx$ (10%) (2) $\int_{-\infty}^{\infty}\exp[-ax^2]\exp(-ikx)dx$ (10%)

【91 北科高分子】【89 高大應化】

【參考解答】

(1) $I=\int_{-\infty}^{\infty}e^{-ax^2}dx=\sqrt{\dfrac{\pi}{a}}$ (2) $J=\int_{-\infty}^{\infty}e^{-ax^2}e^{-jkx}dx=\sqrt{\dfrac{\pi}{a}}e^{\frac{-k^2}{4a}}$

13. Find the Fourier transform of the following function. $f(t)=4e^{-3t^2}\sin(2t)$

【90 台科電機 10%】

【參考解答】 $F[4e^{-3t^2}\sin(2t)]=\dfrac{2}{i}[\sqrt{\dfrac{\pi}{3}}e^{-\frac{1}{12}(\omega-2)^2}-\sqrt{\dfrac{\pi}{3}}e^{-\frac{1}{12}(\omega+2)^2}]$

14. (1)Prove $F[f(x)e^{ax}]=F(\omega-ai)$, where $f(\omega)=F[f(x)]$.

(2)If $g(x)$ is absolutely integrable over $-\infty<x<\infty$, then $F[g(x)]$ exist.

【86 中興環工 20%】

【參考解答】

(1) $F[f(x)e^{ax}]=\int_{-\infty}^{\infty}f(x)e^{i(\omega-ai)x}dx=F(\omega-ai)$

(2) 當 $\int_{-\infty}^{\infty}|g(x)|dx$ 存在，即絕對可積分，則 $F[g(x)]$ 存在，此為 Fourier transform 存在之充分條件。

15. Find $F[1]$, $F[\sin mt]$, $F[\cos mt]$. 【88 台科電子 10%】

【參考解答】

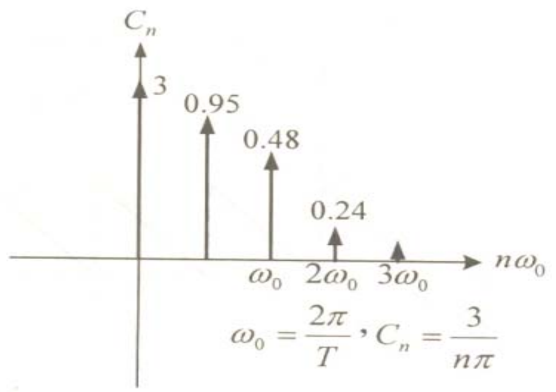
$F[1]=\int_{-\infty}^{\infty}1\bullet e^{-i\omega t}dt=2\pi\delta(\omega)$ ， $F[\sin mt]=\dfrac{1}{2i}[2\pi\delta(\omega-m)-2\pi\delta(\omega+m)]$ ，同理可證：

$F[\cos mt]=\dfrac{1}{2i}[2\pi\delta(\omega-m)+2\pi\delta(\omega+m)]$

16. Find the Fourier transform of the periodic function $f(t)$, of period T, and sketch $f(t)$ and the amplitude spectrum. $f(t)=\frac{3}{T}t,\ 0<t<T$

【90 高雄科大電控 12%】

【參考解答】取 Fourier sine series 展開， $f(t)=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3}{n\pi}\cos(n\omega_0 t+\pi)$



17. Find the "Fourier transform" of the following periodic function whose definition in one period is $f(t)=\begin{cases}0, & -\pi\le t<0\\ \sin(t), & 0\le t<\pi\end{cases}$. 【90 交大機械 25%】

【參考解答】

$$F[f(t)]=2\delta(\omega)+\frac{\pi}{2i}[\delta(\omega-1)-\delta(\omega+1)]+\sum_{n=2,4,6}^{\infty}\frac{-2}{(n^2-1)}[\delta(\omega-n)+\delta(\omega+n)]$$

18. A periodic function whose definition in one period is

$$f(t)=3\sin\frac{\pi}{2}t+5\sin 3\pi t,-2<t<2$$

(1) Find the Fourier series of $f(t)$.

(2) Find the Fourier transform of $f(t)$. 【90 台大機械 15%】

【參考解答】

(1) $f(t)=3\sin\frac{\pi t}{2}+5\sin 3\pi t$

(2) $F[f(t)]=\frac{3\pi}{i}[\delta(\omega-\frac{\pi}{2})-\delta(\omega+\frac{\pi}{2})]+\frac{5\pi}{i}[\delta(\omega-3\pi)-\delta(\omega+3\pi)]$

19. (1) Find the Fourier integral representation of the function $f(x)=\begin{cases}1, & |x|\le 1\\ 0, & |x|>1\end{cases}$.

(2) By using result in (1) to evaluate $\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega$.

(3) Verify your answer in (2) by integrating $\frac{e^{iz}}{z}$ around the contour as shown in following figure and let $r\to 0,\ R\to 0$. 【91 逢甲電機 20%】

【參考解答】

(1) $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}e^{i\omega x}d\omega$ 爲 Fourier integral.

(2) $\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$

(3) $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$

20. Find the Fourier transforms of the following functions:

(1) $f(t)\cos\omega_0 t$ (2) $f(t)\cos\omega_0 t\cos\omega_0 t$ 【90 高雄科大電控 16%】

【參考解答】

(1) $F[f(t)\cos\omega_0 t]=\frac{1}{2}F(\omega+\omega_0)+\frac{1}{2}F(\omega-\omega_0)$

(2) $F[f(t)\cos^2\omega_0 t]=\frac{1}{2}F(\omega)+\frac{1}{4}F(\omega+2\omega_0)+\frac{1}{4}F(\omega-2\omega_0)$

21. Show the following Fourier transform theorems:

(1) convolution theorem $F\{f*g\}=\sqrt{2\pi}F\{f\}F\{g\}$

(2) shifting theorem: $F\{f(x-a)\}=e^{-j\omega a}F\{f(x)\}$

(3) autocorrelation theorem: $F[\int_{-\infty}^{\infty}f(\tau)f(\tau-x)d\tau]=\sqrt{2\pi}\,|F\{f\}|^2$

【88 清大電機 12%】

【參考解答】

(1) 令 $t=x-\tau=\sqrt{2\pi}F[f]\bullet F[g]$

(2) 令 $t = x - a = e^{-j\omega a} \bullet \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \bullet e^{-j\omega t} \bullet dt = e^{-j\omega a} \bullet F[f(x)]$

(3) 令 $\tau - x = t = \sqrt{2\pi} \left| F[f]^2 \right|$ 得證

22. Find the convolution of a rectangular pulse $f(t)$ and triangular pulse $h(t)$

where $\begin{cases} f(t)=1, & |t| \le 1 \\ f(t)=2, & |t| > 1 \end{cases}$, $\begin{cases} h(t)=t, & 0 \le |t| \le 3 \\ h(t)=0, & otherwise \end{cases}$. 【87 成大醫工 20%】

【參考解答】

$$f(t) * h(t) = \frac{1}{2} H(t+1) \bullet (t+1)^2 - \frac{1}{2} H(t-1) \bullet (t-1)^2 + \frac{1}{2} H(t-2) \bullet (8 - t^2 - 2t) - \frac{1}{2} H(t-4) \bullet (8 - t^2 + 2t)$$

23. Determine the Fourier transform of the following functions.

(1) $e^{-3|t|}$ (2) $\frac{5e^{-3|t|}}{t^3 - 4t + 13}$ 【90 雲科電機 10%】

【參考解答】

(1) $F[e^{-3|t|}] = \int_0^{\infty} e^{-3t} \bullet 2\cos\omega t dt = \frac{6}{9 + \omega^2}$

(2) 當 $3 - \omega \ge 0$，$\int_{-\infty}^{\infty} \frac{5e^{i3t} \bullet e^{-i\omega t}}{t^2 - 4t + 13} dt = 2\pi i \frac{5e^{i(3-\omega)(-2+3i)}}{6i}$

當 $3 - \omega < 0$，$\int_{-\infty}^{\infty} \frac{5e^{i3t} \bullet e^{-i\omega t}}{t^2 - 4t + 13} dt = -2\pi i \frac{e^{i(3-\omega)(-2-3i)}}{-6i}$

24. Let $z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$,

(1) Prove the area under $z(t)$ is the product of the area under $x(t)$ and $y(t)$ over the interval $-\infty < t < \infty$.

(2) Given an interpretation.

【參考解答】取 $\omega = 0$，$\int_{-\infty}^{\infty} z(t) dt = \int_{-\infty}^{\infty} x(t) dt \bullet \int_{-\infty}^{\infty} y(t) dt$。

25. Let $x(t) \leftrightarrow X(i\omega)$, $y(t) \leftrightarrow Y(i\omega)$, and $z(t) \leftrightarrow Z(i\omega)$ denote Fourier transform

pairs, related by $Z(i\omega)=\int_{-\infty}^{\infty} z(t)e^{j\omega t}dt$, $Z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} z(j\omega)e^{+j\omega t}d\omega$ if $Z(j\omega)=x(j\omega)y(j\omega)$, express $z(t)$ in terms of $x(t)$ and $y(t)$.

【89 中山電機 10%】

【參考解答】$z(t)=X(t)*Y(t)=\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$

■ Fourier transform 解 O.D.E

1. Find the particular solution of the differential equation $y''+cy'+y=r(t)$, with $c>0$ and $r(t)$ given as $r(t)=\frac{t}{12}(\pi^2-t^2)$ if $-\pi<t<\pi$ and $r(t+2\pi)=r(t)$.

【91 成大土木 20%】

【參考解答】$y_p=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\frac{(1-n^2)\sin nt-nc\cos nt}{(1-n^2)^2+c^2n^2}$

2. Find Fourier series solution of $\frac{d^2T}{dx^2}-T=-\delta(x-a),\ 0<x<1$,

$\frac{dT(0)}{dx}=\frac{dT(1)}{dx}=0$

where δ is the Dirac delta function, a is a constant and $0<a<1$.

【90 交大機械 20%】

【參考解答】$T=1+\sum_{n=1}^{\infty}\frac{2\cos ant}{n^2\pi^2+1}\cos n\pi x$

3. Find the steady-state solution $y(t)$ of $y''+0.02y'+25y=r(t)$, where

$r(t)=\begin{cases} t+\frac{\pi}{2}, & if\ -\pi<t<0 \\ -t+\frac{\pi}{2}, & if\ 0<t<\pi \end{cases}$, and $r(t+2\pi)=r(t)$. 【91 雲科電機 10%】

【參考解答】

$y_p=\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\frac{(25-n^2)\cos nt+0.02n\sin nt}{(25-n^2)^2+(0.02n)^2}$ as $t\to\infty$, $y=y_p$ 如上所示

4. Find the general solution of the differential equation $y''+\omega^2 y=r(t)$,

$r(t)=\begin{cases} t+\pi, & if\ \ -\pi<t<0 \\ -t+\pi, & if\ \ 0<t<\pi \end{cases}$, and $r(t+2\pi)=r(t)$, $\omega\neq 1,2,3$【91 中央化工 20%】

【參考解答】通解 $y=c_1\cos\omega t+c_2\sin\omega t+\dfrac{\pi}{2\omega^2}+\sum_{n=1,3,5}^{\infty}\dfrac{4}{n^2\pi}\dfrac{1}{\omega^2-n^2}\cos nt$.

5. Find a formal Fourier series solution of the endpoint value problem.

$$x''+4x=4t\ ;\ \ x(0)=1,\ \ x(1)=0$$

【89 交大電子 10%】

【參考解答】 $x=2\sum_{n=1}^{\infty}\dfrac{1}{4-n^2\pi^2}[\dfrac{4(-1)^{n+1}}{n\pi}-n\pi]\bullet\sin n\pi t$

6. 已知一微分方程式 $y''+5y'+6y=f(x)$，其中 $f(x)=\begin{cases} b, & -a\le x\le a \\ 0, & x<-a\ \ and\ \ x>a \end{cases}$

(1) 試以傅立葉積分(Fourier Integral)展開 $f(x)$。

(2) 試求解此微分方程式。 【91 雲科營建 20%】

【參考解答】

(1) $f(x)=F^{-1}[F(w)]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{2b}{\omega}\sin\omega a\bullet e^{i\omega x}d\omega$

(2) $f[y'']=-\omega^2\bar{y}$ ， $F[y']=i\omega\bar{y}$ ， $\bar{y}=F[y]$

$$y=u(x+a)\{\frac{b}{6}+\frac{b}{2}[2e^{-2(x+a)}]+\frac{b}{3}[e^{-3(x+a)}-2e^{-3(3-a)}]\}-\frac{b}{6}u(x-a)+bu(x-a)\bullet[-\frac{1}{2}e^{-2(x-a)}+\frac{1}{3}e^{-3(x-a)}]$$

7. (1) What are the conditions under which a Fourier series representation for a given function $f(t)$ is possible?

(2) Solve $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+y=f(t)$, if $f(t)=\begin{cases} 3, & t^2<1 \\ 0, & t^2>1 \end{cases}$. 【90 中原醫工 15%】

【參考解答】

(1)當 $f(t)$為週期函數，Fourier series 存在。

(2)利用留數積分，得 $y(t)=\begin{cases} 0, & t\le 1 \\ 3-3(t+2)e^{-(t+1)}, & -1<t\le 1 \\ -3(t+2)e^{-(t-1)}+3te^{-(t-1)}, & 1<t \end{cases}$ 。

8. Solve the following first order differential equation by applying the Fourier transform.

$$y' - 2y = H(t)e^{-2t}, -\infty < t < \infty$$

where $H(t)$ is the unit step function (Heaviside function). 【89 台科電機 10%】

【參考解答】$\bar{y} = -\dfrac{1}{\omega^2 + 4}$ ， $y = -\dfrac{1}{4}e^{-2|t|}$

■ Fourier transform 解 P.D.E.

1. Find the solution of the wave equation corresponding to the triangular initial defection and initial velocity zero.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \bullet \frac{\partial^2 u}{\partial x^2}, \quad u_t(x,0) = 0, \quad u(0,t) = 0, \quad u(\ell,t) = 0$$

$$u(x,0) = \begin{cases} \dfrac{2k}{\ell}x, & 0 < x < \dfrac{1}{2}\ell \\ \dfrac{2k}{\ell}(\ell - x), & \dfrac{1}{2}\ell < x < \ell \end{cases}$$

【91 元智機械 20%】

【參考解答】$u = \sum_{n=1}^{\infty} \dfrac{8k}{n^2\pi^2} \sin\dfrac{n\pi}{2} \cos\dfrac{cn\pi t}{\ell} \bullet \sin\dfrac{n\pi t}{\ell}$

2. Solve $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$, with $u(x=0,t) = u(x=3,t) = 0$ for all t, and $u(x,t=0) = \sin(14\pi x)$, $u_i(x,0) = f(x)$. Derive a complete solution.

【91 清大電機 15%】

【參考解答】$u = \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n}\int_0^3 f(x)\sin\dfrac{n\pi x}{3}dx \sin\dfrac{n\pi t}{3} \sin\dfrac{n\pi x}{3} + \cos 14\pi t \bullet \sin 14\pi x$

3. Slove $\dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x}$, $0 < x < L$, $t > 0$ $u(0,t) = u(L,t) = 0$, $u(x,0) = L[1 - \cos\dfrac{2\pi x}{L}]$.

【91 清大微機電 25%】

【參考解答】$u = \sum_{n=1,3,5}^{\infty} \dfrac{L}{\pi} \bullet \dfrac{-16}{n(n^2-4)} e^{-3(\frac{n\pi}{L})^2 t} \bullet \sin\dfrac{n\pi x}{L}$

4. Solve the nonhomogeneous heat equation shown below: $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\sin(\pi x)$.

Boundary conditions: $u(0,t)=u(1,t)=0 \quad 0<t<\infty$

Initial condition: $u(x,0)=\sin(2\pi x)$ 【89 中正機械 15%】

【參考解答】 $u=\frac{1}{\pi^2}[1-e^{-\pi^2 t}]\sin\pi x+e^{-4\pi^2 t}\bullet\sin 2\pi x$

5. Consider the Laplace's equation in polar coordinates $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$.

Find a solution $u(r,\theta)$ of Laplace's equation inside a region $r\le a,\ 0\le\theta\le a$ that satisfies the boundary conditions $u(r,0)=u(r,a)=0, u(a,\theta)=k$.

【91 交大電子 15%】

【參考解答】 $u(r,\theta)=\sum_{n=1,3,5}^{\infty}\frac{4k}{n\pi}(\frac{r}{a})^{\frac{n\pi}{a}}\bullet\sin\frac{n\pi\theta}{\alpha}$

6. Solve the partial differential equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0\ (0<x<a,\ 0<y<b)$.

With the corresponding boundary conditions $f(x,0)=f(x,b)=0\ (0<x<a)$

$f(0,y)=0,\ f(a,y)=$ A constant $(0<y<b)$. 【91 台科化工 15%】

【參考解答】 $f=\sum_{n=1,3,5}^{\infty}\frac{4A}{n\pi}\frac{1}{\sin\frac{n\pi a}{b}}\sinh\frac{n\pi x}{b}\bullet\sin\frac{n\pi y}{b}$

7. Solve the partial differential equations: $C\frac{\partial^4 v}{\partial x^4}+E\frac{\partial^2 v}{\partial t^2}=0,\ for\ t\ge 0;\ 0\le x\le L$.

In which $v=v(x,t)$, C and E are constants, given that the initial and boundary conditions are

At $t=0$: $v(x,0)=v_0,\ \frac{\partial v(x,0)}{\partial t}=v_0$

At $x=0$: $v(0,t)=0,\ \frac{\partial^2 v(0,t)}{\partial x^2}=0$

At $x=L$: $v(L,t)=0,\ \frac{\partial^2 v(L,t)}{\partial x^2}=0$ 【91 交大機械 25%】

【參考解答】 $v=\sum_{n=1}^{\infty}\frac{2}{L}b_n\sin\frac{n\pi x}{L}$

8. Consider the following boundary/initial value problem:

Equation $U_{xx}=U_{tt}+U_t$

B.C. $U=0$ at $x=0$ $U=0$ at $x=2$

I.C. $U=f(x)$ when $t=0$

$U_t=g(x)$ when $t=0$

Here U_{xx}, U_t ,and U_n are partial derivatives of U. If the solution is expressed as $U=\sum_{n=1}^{\infty}F_n(x)G_n(t)$ please find out the expression of $G_n(t)$.

【90 中央土木 25%】

【參考解答】 $G_n=b_n=c_1e^{-\frac{t}{2}}\cos\frac{\sqrt{n^2\pi^2-1}}{2}t+c_2e^{-\frac{t}{2}}\sin\frac{\sqrt{n^2\pi^2-1}}{2}t$

9. Suppose a laterally insulated long thin bar with length L and of constant cross section and homogeneous material is oriented along x-axis. The temperature $u(x,t)$ of the bar satisfies the following 1-D heat equation: $u(x,t)=c^2u_{xx}(x,t)$

Find the temperature of the bar for any time $t>0$ if the ends of the bar are kept at different constant temperatures $u(x,0)=U_t$ and $u(L,t)=U_2$ and initially $u(x,0)=f(x)$. 【90 清大電機、電子 10%】【90 淡江化工 25%】

【參考解答】 $T=\omega+(100-x)$

10. 試求解下列的偏微分方程： $k\frac{\partial^2 u}{\partial x^2}+r=\frac{\partial u}{\partial t}$ ， $0<x<1,\ t>0$

邊界條件： $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

初値條件： $u(x,0)=f(x),\ 0<x<1$ ，其中 $k,\ r$ 和 u_0 均爲常數

【91 台大生機 10%】

【參考解答】 $u=\omega-\frac{r}{2k}(x^2-x)+u_0x$

11. Solve the following nonhomogeneous heat equation $\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=e^{-ax},\ 0<x<L$,

$u(0,t)=u(L,t)=0,\ u(x.0)=f(x)$ 【91 中原土木 20%】

【參考解答】$u=\omega(x,t)+v(x)$

12. 請求解 $u_{tt}-u_{xx}=0$ for $0<x<1,\ t>0$ $u(x,0)=1$ for $0\le x\le 1$

$\frac{\partial u}{\partial t}(x,0)=\sin^3 \pi x$ for $0\le x\le 1$ $\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=0$

求出 $u(\frac{1}{2},2)=?$ 【89 淡江環工 25%】

【參考解答】as $t=2,\ x=\frac{1}{2},\ u=1-\frac{8}{3\pi}$

13. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<a,\ 0<y<b$.

$u_x(0,y)=0,\ u_x(a,y)=0,\ u(x,0)=0,\ u(x,b)=1$ 【90 雲科機械 25%】

【參考解答】$u=A_0=\frac{1}{b}y$

14. Slove the partial differential equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ 0<x<\pi,\ t>0$, I.C.

$u(x,0)=x$, and B.C. $\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t)=0$. 【91 中興化工 20%】

【參考解答】$u=\frac{\pi}{2}+\sum_{n=1,3,5}^{\infty}\frac{-4}{n^2\pi}e^{-n^2 t}\cos nx$

15. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\ (x>0,y>0)$, $u(0,y)=0,\ (y>0)$, $u(x,0)=\begin{cases}4, & 0\le x\le 2\\ 0, & x>2\end{cases}$.

【89 淡江電機 20%】

【參考解答】$u=\frac{8}{\pi}\int_0^{\infty}\frac{1-\cos 2\omega}{\omega}e^{-\omega y}\bullet\sin\omega x d\omega$

16. Slove the boundary value problem using Fourier Transform in x .

$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}(-\infty<x<\infty,t>0)$ $u(x,0)=f(x),(-\infty<x<\infty)$

【91 暨南土木 15%】【90 台大電機 7%】

【參考解答】 $u=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} f(z)e^{-\frac{(z-x)^2}{4t}}dz$

17. Slove $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial^2 x}$, $U_x(0,t)=0$, $U(x,0)=x$ if $0<x<1$ and if $x>1$, $U(x,t)$ is bounded where $x>0,\ t>0$. 【91 中興材料 20%】

【參考解答】 $u=\frac{2}{\pi}\int_0^{\infty}\frac{\omega\sin\omega-1+\cos\omega}{\omega^2}e^{-\omega^2 t}\bullet\cos\omega x d\omega$

18. By using Fourier transform, slove $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}+\delta(x)\delta(t)$,

$u(x,0)=\delta(x),\ \lim_{x\to\pm\infty}u(x,t)=0$. 【91 海洋機械 20%】

【參考解答】 $u=\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}e^{-\frac{x^2}{4t}}+\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}e^{-\frac{x^2}{4t}}\bullet H(t)$

19. Slove following partial differential equation by Fourier Transform.

$\frac{\partial^2 u(x,t)}{\partial t^2}=9\frac{\partial^2 u(x,t)}{\partial x^2},\ (-\infty<x<\infty,\ t>0)$

$u(x,0)=4e^{-5|x|},\ \frac{\partial u(x,0)}{\partial t}=0\ (-\infty<x<\infty)$ 【90 逢甲電機 15%】

【參考解答】 $u=2e^{-5|x-3t|}+2e^{-5|x+3t|}$

20. Slove PDE by Fourier transform $\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+tu=0\ (x>0,t>0)$,

$u(x,0)=xe^{-x},\ u_x(0,t)=0$ 【90 海洋光電 15%】

【參考解答】 $u=\frac{2}{\pi}\int_0^{\infty}[\frac{2}{(\omega^2+1)^2}-\frac{1}{\omega^2+1}]e^{-\omega^2 t-\frac{1}{2}t^2}\bullet\cos\omega x d\omega$ 爲所求。

21. Consider the problem of determining the temperature distribution in a bar

extending from zero to infinity if the left end is kept at zero temperature and the initial temperature in the cross-section at x is $f(x)$, where

$f(x)=\begin{cases}\pi - x, & 0 \le x \le \pi \\ 0, & x \ge \pi\end{cases}$. Slove the problem as the mathematical model is

$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ $(x>0, t>0)$, a is constant, t is the time. 【89 台科電子 14%】

【參考解答】$u=\frac{2}{\pi}\int_0^{\infty}(\frac{\pi}{\omega}-\frac{1}{\omega^2}\sin\omega\pi)e^{-a^2\omega^2 t}\bullet\sin\omega t d\omega$

22. A semi-infinite thin bar $x \ge 0$ whose surface is insulated has an initial temperature equal to $f(x)$. A temperature of zero is suddenly applied to the end $x=0$ and maintained.

(1) Set up the boundary-value problem for the temperature $u(x,t)$ at any point x at time t

(2) Slove (1). 【90 中興土木 20%】

【參考解答】$u=\frac{1}{2\pi}\int_0^{\infty}f(z)\frac{1}{\sqrt{\alpha t}}[e^{-\frac{(z-x)^2}{4\alpha t}}-e^{-\frac{(z+x)^2}{4\alpha t}}]dz$

23. Please solve the following partial differential equation $\frac{\partial y}{\partial t}=\alpha\frac{\partial^2 y}{\partial x^2}$ subject to the initial and boundary conditions $y(x,0)=y_0$, $y(0,t)=0$ and $y(\infty,t)=y_0$.

(Note: $\int_0^{\infty}e^{-a\lambda^2 t}\cos\lambda x d\lambda=\frac{1}{2}(\frac{\pi}{at})^{1/2}e^{-x^2/4at}$)

【91 清大動機 10%】【90 成大環工 20%】

【參考解答】$u=y_0 erf(\frac{x}{2\sqrt{at}})$

24. Solve the partial differential equation by Fourier sin transformation. $u-u_{xx}=0$ for $0<x<\infty$, $t>0$, $u(0,t)=g(t)$, $u(x,0)=0$, and $u(x,t)$ is bounded.

【89 台科機械 20%】

【參考解答】$u=\frac{x}{2\sqrt{\pi}}\int_0^{t}g(\tau)\bullet\frac{1}{(t-\tau)^{3/2}}e^{-\frac{x^2}{4(t-\tau)}}\bullet d\tau$

25. Please solve the following partial differential equation as $u_t = u_{xx} + u_{yy}$ where

$0 \le t < \infty,\ 0 \le x \le \pi,\ 0 \le y \le \infty$ initial condition $u(x,y,0) = 0,\ |u(x,y,t)| < M$

(bounded) boundary condition $u(0,y,t) = 0,\ u(\pi,y,t) = 0,\ u(x,0,t) = 100$.

【89 北科機電整合 20%】

【參考解答】$u = \dfrac{4}{\pi^2}\sum_{n=1}^{\infty}\int_0^{\infty}\dfrac{1-(-1)^n}{n^2+\omega^2}\dfrac{100\omega}{n}[1-e^{-(n^2+w^2)}]\sin\omega y d\omega \sin nx$

■ 分離變數法(separation of variable)

1. We wish to slove the Laplace's equation using separation of variables $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ within $0 \le x \le b$ and $0 \le y \le a$ with boundary values given by $u(x=0, 0 \le y \le a) = u(x=b, 0 \le y \le a) = u(0 \le x \le b,\ y=0) = 0$ and $u(0 \le x \le b, y = a) = 1$. Let $u(x,y) = X(x)Y(y)$.
   (1) Show that $X''/X = \lambda$,where λ is constant.
   (2) Derive the boundary conditions for $X(0)$ and $X(b)$.
   (3) Discuss whether $\lambda > 0, \lambda = 0$ or $\lambda < 0$.
   (4) Impose the boundary conditions for X to determine the possible values of λ

【91 中山通訊 20%】

【參考解答】 $u = \sum_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{b} \bullet \sinh\dfrac{n\pi y}{b}$，代入$u(x,a) = 1$，

$1 = \sum_{n=1}^{\infty} B_n \bullet \sinh\dfrac{n\pi a}{b} \bullet \sin\dfrac{n\pi x}{b}$ 得 $B_n = \dfrac{1}{\sinh\dfrac{n\pi a}{b}}\dfrac{2}{n\pi}\bullet[1-(-1)^n]$

2. 函數$u(x,t)$滿足一維擴散方程式$\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ 依下步驟在$0 < x < 4$區間內，求解函數$u(x,t)$，其邊界條件爲$u(0,t) = u(4,t) = 0$。
   (1) 說明此方程式是 separable。並寫出變數 x 和 t 的個別方程式
   (2) 滿足邊界條件的通解爲何？
   (3) 若已知時間 t=0 時函數爲$u(x,0) = -\sin(\pi t) + \sin(2\pi x)$，求$u(x,t)$。

【91 中央光電 20%】

【參考解答】

(1) $\begin{cases} f''-\lambda f=0, & f(0)=0 \\ T'-4\lambda T=0, & f(4)=0 \end{cases}$ ，P.D.E 可變分離

(2)P.D.E.通解 $u=fT=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{4} e^{-(\frac{n\pi}{2})^2 t}$

(3) $u=-1\sin\pi x e^{-4\pi^2 t}+\sin 2\pi x e^{-16\pi^2 t}$

3. Consider the heat-conduction-like partial differential equation (PDE) for $u(x,t)$: $\frac{\partial u(x,t)}{\partial t}=t\frac{\partial^2 u(x,t)}{\partial x^2}$ with boundary conditions $u(0,t)=0,\ u(L,t)=0$, initial condition $u(x,0)=f(x)$.
   (1) By assuming that the solution can be written as $u(x,t)=X(x)T(t)$, show that $T(t)$ and $X(x)$ must satisfy $T'(t)+\lambda tT=0$ and $X''(x)+\lambda X=0$, where λ = constant, and $X(0)=0,\ X(L)=0$.
   (2) Show that to satisfy $X(0)=0,\ X(L)=0$, λ must be positive and find the solution $X(t)$ and the eigenvalue λ.
   (3) Now that λ is known, slove for $T(t)$.
   (4) What is the general solution to the P.D.E ? 【91 交大光電 30%】

【參考解答】

(1) $T'-\lambda tT=0$

(2)特徵函數 $X_n(x)=c_n\sin\frac{n\pi}{L}x$

(3) $T=ke^{-\frac{1}{2}(\frac{n\pi}{L})^2 t^2}$

(4) $u=\frac{2}{L}\sum_{n=1}^{\infty}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \bullet e^{-\frac{1}{2}(\frac{n\pi}{L})^2 t^2} \bullet \sin\frac{n\pi x}{L}$

4. A vertical cross section of a long high wall 30cm thick has the shape of the semi-infinite strip $0<x<30,\ y<0$. The face $x=0$ is held at temperature zero, while the face x=30 is insulated. Given temperature $r(x,0)=25$, find the steady-state temperature within the wall. 【91 交大電信 15%】

【參考解答】 $u=\sum_{n=1}^{\infty}\frac{100}{(2n-1)\pi}e^{-\frac{2n-1}{60}\pi y} \bullet \sin\frac{2n-1}{60}\pi x$

5. Using the method of separating variables to solve the boundary-value problem of

the one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t)=0$ and $u(L,t)=0$ for all t, where $u(x,t)$ is the deflection of string and L is the length of the string. 【91 台師大光電 20%】

【參考解答】取 $u_t(x,0)=g(x),\ g(x)=\sum_{n=1}^{\infty}\frac{cn\pi}{L}B_n\sin\frac{n\pi x}{L}$

$$B_n=\frac{L}{cn\pi}\bullet\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

6. 一細長桿長度 ℓ，表面絕熱，初始溫度 100℃，設左端亦絕緣右端保持恆溫 $u(\ell,t)=0$℃，求溫度 $u(x,t)$。Hint：heat equation $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$，$u_x(0,t)=0,\ u(\ell,t)=0$。 【91 中興土木 25%】

【參考解答】代入 $u=(x,0)=100=\sum_{n=1}^{\infty}A_n\bullet\cos\frac{2n-1}{2\ell}\pi x$

$$A_n=\frac{2}{\ell}\int_0^{\ell}100\bullet\cos\frac{2n-1}{2\ell}\pi x dx=\frac{400}{(2n-1)\pi}(-1)^{n+1}$$

7. A boundary value problem is shown as follow: $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\ (0<x<L,\ t>0)$

$u(0,t)=0,\ \frac{\partial u}{\partial x}(L,t)=-Au(L,t)\ (t\ge 0)\ (A>0),\ u(x,0)=f(x)\ (0<x<L)$.

【90 台科電子 15%】

【參考解答】取 $t=0,\ f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{\alpha_n}{L}x$

$$B_n=\frac{\int_0^L f(x)\sin\frac{\alpha_n x}{L}dx}{\int_0^L\sin^2\frac{\alpha_n x}{L}dx}=\frac{\int_0^L f(x)\sin\frac{\alpha_n x}{L}dx}{\frac{L}{2}-\frac{L}{4\alpha_n}\sin 2\alpha_n}$$

8. Solve the following problem by the method of separation variables:

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t^2},\ 0\le x\le a,\ 0\le y\le b,\ t\ge 0,$

$u(0,y,t)=0,\ u(a,y,t)=0,\ u(x,0,t)=0,$

$$u(x,b,t)=0,\ u(x,y,0)=1,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=1.$$ 【91 成大造船 20%】

【參考解答】取 $u(x,y,0)=1$，$1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$,

$$B_{mn}=\frac{1}{\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}}\frac{4}{mn\pi^2}[1-(-1)^m][1-(-1)^n]$$

■ 以極座標解 P.D.E.

1. Find the steady state temperature for a thin disk of radius R if the temperature on the boundary is $f(\theta)=\cos^2\theta,\ -\pi<\theta<\pi$. 【91 北科冷凍 20%】

【參考解答】$T=A_0+A_2r^2\cos 2\theta=\frac{1}{2}+\frac{1}{2R^2}r^2\cos 2\theta$

2. Find the function $f(x,y)$ satisfying the Laplace equation $\nabla^2 f=\frac{\partial^2 f}{\partial^2 x}+\frac{\partial^2 f}{\partial^2 y}=0$ for $x^2+y^2=a,\ a>0$ and the boundary condition $f(x,y)=x^3$ for $x^2+y^2=a$. 【91 中山光電 20%】

【參考解答】$f=A_1r\cos\theta+A_3r^3\cos 3\theta=\frac{3}{4}ar\cos\theta+\frac{1}{4}r^3\cos 3\theta$

3. Let $u(p,\phi)$ denote the steady temperature in a long solid cylinder $a\le\rho\le b,\ -\infty<z<\infty$ when the temperature of the inner surface $\rho=a$ is a given function $f(\phi)=A+B\sin\phi$ where A and B are constants; and temperature of the outer surface $\rho=b$ is zero. Then the governing equation can be written as follows in a cylindrical coordinate. $\rho^2\frac{\partial^2 u(\rho,\phi)}{\partial^2\rho}+\rho\frac{\partial u(\rho,\phi)}{\partial\rho}+\frac{\partial^2 u(\rho,\phi)}{\partial^2\phi}=0$

Please calculate $u(\rho,\phi)$. 【91 北科土木 25%】

【參考解答】$u=(A_0+B_0\ln\rho)+(E_1\rho+F_1\rho^{-1})\sin\phi$

4. Consider the problem of vibrations in a circular membrane of radius a. Let $u(r,t)$ denote the vertical displacement of the membrane, and if the initial conditions are circularly symmetric, then the mathematical formulation of the problem is as follows; $\frac{\partial^2 u}{\partial t^2}=\frac{c^2}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})$, $t>0$, $r<a$, $u(a,t)=0$, $u(r,0)=f(r)$, $\frac{\partial u}{\partial t}(r,0)=g(r)$. Slove $u(r,t)$ in terms of $f(r)$, $g(r)$, where c is a constant.

【91 中山海下技術】

【參考解答】代入 $u_t(r,0)=g(r)=\sum_{n=1}^{\infty}\frac{c\alpha_n}{a}B_nJ_0(\frac{\alpha_n r}{a})$ $B_n=\frac{a}{c\alpha_n}\frac{\int_0^a rg(r)J_0(\frac{a_n r}{a})dr}{\int_0^a rJ_0^{\,2}(\frac{a_n r}{a})dr}$

5. Find the solution of the following partial differential equation.

$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial\xi^2}+\infty\frac{1}{\xi}\frac{\alpha\phi}{\partial\xi}$, $0\le\xi\le 1$, $t\ge 0$, $\phi=1-\xi^2$, $t=0$, $\phi=finite$, $\xi=0$, $\phi=0$, $\xi=1$.

【91 北科化工 20%】

【參考解答】：代入 $\phi(\xi,0)=1-\xi^2$, $1-\xi^2=\sum_{n=1}^{\infty}B_nJ_0(\alpha_n\xi)$,

$$B_n=\frac{\int_0^1\xi(1-\xi^2)J_0(\alpha_n\xi)d\xi}{\int_0^1\xi\bullet J_0^{\,2}(\alpha_n\xi)d\xi}$$

■ 非齊性 P.D.E. (特徵函數展開法)

1. Slove the problem

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$, $0<x<1$, $0<t$

$u(0,t)=1$, $0<t$

$\frac{\partial u}{\partial x}=0, x=1$, $0<t$

$u(x,0)=2$, $0<x<1$

【91 台大化工 10%】

【參考解答】 $u=\omega+1$

2. A temperature distribution $T(x,y)$ at steady state satisfies the Laplace equation

$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$. If the boundary conditions are given as

$$\begin{array}{l} T(x,0)=0,\ T(x,h)=f(x) \\ T(0,y)=0,\ \dfrac{\partial T(\omega,y)}{\partial x}=c \end{array}.$$

Solve $T(x,y)$ for $\begin{cases} c=0 \\ c\neq 0 \end{cases}$. 【91 交大機械 20%】

【參考解答】

當 $c=0$，$T=\frac{2}{\omega}\sum_{n=1}^{\infty}\frac{\sinh\frac{2n-1}{2\omega}\pi y}{\sinh\frac{2n-1}{2\omega}\pi h}\int_0^{\omega} f(x)\sin\frac{2n-1}{2\omega}\pi x dx\bullet\sinh\frac{2n-1}{2\omega}\pi x$

當 $c\neq 0$，$T=u+cx=\frac{2}{\omega}\sum_{n=1}^{\infty}b_n\sin\frac{2n-1}{2\omega}\pi x+cx$

3. By the method of separation of variables, find the solution $u(x,y)$ of the Poisson equation $u_{xx}+u_{yy}=\cos(\pi y)$, in the semi-infinite strip

$0\le x<\infty,\ 0\le y\le 1$, such that $u(0,y)=y,\ u_y(x,0)=u_y(x,1)=0$.

【91 中央機械 25%】

【參考解答】$u=-\frac{1}{\pi^2}\cos\pi y+\omega(x,y)$

4. Solve the following initial-boundary0valued problem of $u(x,t)$.

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2},\ 0<x<1,\ t>0 \\ u(0,t)=t,\ t>0 \\ u(1,t)=1,\ t>0 \\ u(x,0)=x,\ 0<x<1 \end{cases}$$

【89 台大應力 25%】

【參考解答】$u=\omega+t-(t-1)x$ 可得

5. (1) Solve the boundary value problem $\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = 9\dfrac{\partial^2 y}{\partial x^2} + x^2 \quad (0 < x < 4,\ t > 0), \\ y(0,t) = y(4,t) = 0 \quad (t > 0), \\ y(x,0) = 0 \quad (0 < x < 4), \\ \dfrac{\partial y}{\partial t}(x,0) = 0 \quad (0 < x < 4) \end{cases}$

(2) Discuss in detail the characteristics of eigen values and eigen functions of the above partial differential equation. 【91 中興土木 25%】

【參考解答】$y = u + \dfrac{1}{108}(64x - x^4)$

■ 座標轉化與重疊原理

1. Solve $\begin{cases} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ 0 \le x \le \ell,\ 0 \le y \le \ell \\ T(0,y) = 0,\ T(x,0) = 0 \\ T(x,\ell) = f(x),\ T(\ell,y) = g(y) \end{cases}$ 【90 台大工程科學 15%】

【參考解答】$T(x,y) = \dfrac{2}{\ell}\sum_{n=1}^{\infty}\dfrac{1}{\sinh n\pi}\int_0^{\ell} g(y)\sin\dfrac{n\pi y}{\ell}dy \bullet \sin\dfrac{n\pi x}{\ell}\sin\dfrac{n\pi y}{\ell}$

2. Solve the following partial differential equation.

$$\begin{cases} \nabla^2 u(x,y) = 0,\ 0 < x < a,\ 0 < y < b \\ u(0,y) = 0,\ u(a,y) = y \\ u(x,0) = 0,\ u(x,b) = x \end{cases}$$

【89 成大土木 25%】

3. Solve the following boundary-value problem $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ 0 \le x \le 1,\ 0 \le y \le 1 \\ u(x,0) = 1 \\ u(1,y) = 0 \\ u(x,1) = 0 \\ u(0,y) = 1 \end{cases}$

【90 成大造船 20%】

■ 一階 P.D.E 與其解間之關係

1. Solve $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$, $u(x,0) = \sin x$. 【91 海洋河工 15%】

【參考解答】 $u = \sin(x+t)$

2. 請解 Partial Equation $\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$ 。 【91 中央光電 10%】

【參考解答】 $u = c_2$ ，取 $q = u$ ，得 PDE 通解 $u = f(ye^x)$

3. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2u = 0$, $u(x,0) = \sin x$. 【91 北科通訊 15%】

【參考解答】 $u = e^{-2y}\sin(x-y)$

4. Solve following PDE with boundary condition $A(0,t) = A_0$,

$(\frac{\partial}{\partial x} + \frac{n}{c}\frac{\partial}{\partial t})A(x,t) = i\beta\sin(\omega t - kx)\bullet A(x,t)$. Here x and t are variable, c, n, i, β,

ω, A_0 are constants $i = \sqrt{-1}$. 【90 元智電機微波、光電組 20%】

【參考解答】 $f(y) = A_0\bullet\exp[\frac{-i\beta}{k - \frac{n}{c}\omega}\cos(\frac{\omega}{c}y)]$

5. Solve the system $\frac{dx}{x^2+y^2-yz} = \frac{dy}{-x^2-y^2+xz} = \frac{dz}{(x-y)z}$.

【88 交大電子 7%】

【參考解答】 $\begin{cases} x^2+y^2 = C_1 z^2 \\ x+y-z = C_2 \end{cases}$

6. Solve $(y+z)\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-(x-y)=0$ with condition $y=1,\ z=1+x$.

【92 淡江環工 20%】

【參考解答】 $x^2-(y+z)^2=-2(\frac{x+z}{y})-2$ 爲 PDE 之解

■ 常係數 P.D.E.

1. The vertical displacement $u(x,t)$ of an infinitely long string is determines from the initial-value problem：

$$\frac{\partial^2 u}{\partial t^2}=C^2\frac{\partial^2 u}{\partial x^2},\ -\infty<x<\infty,\ u(x,0)=f(x),\ \frac{\partial u(x,0)}{\partial t}=g(x).$$

(1) Find the D'Alembert solution of $u(x,t)$.

(2) If $f(x)=\sin(x), g(x)=1$, find $u(x,t)$.

【91 逢甲土木 20%】【91 成大工程科學 20%】

【參考解答】

$$(1)\ u=\frac{1}{2}[f(x-Ct)+f(x+Ct)]+\frac{1}{2c}\int_{x-Ct}^{x+Ct}g(x)dx$$

$$(2)\ u=t+\frac{1}{2}[son(x-Ct)+\sin(x+Ct)]$$

2. Consider a wave equation of $u(x,t)$, $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}\ (-\infty<x<\infty,\ 0<t)$,

if the initial conditions are given $u(x,0)=\begin{cases}\cos(x), & -\pi<x<\pi\\ 0, & otherwise\end{cases}$, $\frac{\partial u}{\partial t}(x,0)=0$.

Find and graph the waveform of $u(x,t)$ at $t=3.0$. 【91 台大工程科學 20%】

【參考解答】 $u=\begin{cases}\frac{1}{2}\cos(x-1), & -\pi\le x-t\le\pi\\ \frac{1}{2}\cos(x+1), & -\pi\le x+t\le\pi\\ 0, & elsewhere\end{cases}$

3. Using the indicated transformations, solve the following equation.

$u_{xx} = u_{yy}$ $(v = y + x,\ z = y - x)$ 【91 暨南電機】

【參考解答】$u = f_1(y+x) + f_2(y-x)$

4. Slove $u_{xx} - 4u_{xy} + 3u_{yy} = 0$ by D'alembert's method; that is, change independent variables and reduce the equation into a simplified form(the normal form), and then write down the general solution.

【91 清大電機 10%】

【參考解答】$u = f_1(y+3x) + f_2(y+x)$

5. Solve the partial differential equation $2u_x - 3u_y + 2u = 2x$, where the initial condition $u(x,y) = x^2$ for the line $2y + x = 0$. 【88 台科控制 20%】

【參考解答】$u = e^{y+\frac{1}{2}x}[\frac{1}{4}(2y+3x)^2 - \frac{1}{2}(2y+3x)+1] + x - 1$

6. For one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, find $u(x,t)$ of the string of length π. The initial velocity is zero, and the initial deflection is $\sin 3x$. Please show that the solution is of form $u = \sum f(a \pm t)$. State the physical meanings for your solution. 【90 中央物理 15%】

【參考解答】$x + t = c_1$ 與 $x - t = c_2$ 為 u 之特徵曲線。

7. 試將 Laplacian equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ 轉換成極座標形式，其中 $x = r\cos\theta,\ y = r\sin\theta$。 【90 中興土木 25%】

【參考解答】$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$