習題演習

向量分析

習題演習：向量分析

■ 向量的基本運算

【習題 1】

Tow vector $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ in R^4 span a subspace E, where $\overrightarrow{v_1}=\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$, $\overrightarrow{v_2}=\begin{bmatrix}1\\9\\5\\-3\end{bmatrix}$.

a. Perform the Gram-Schmidt process to find an orthonormal basis $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ of E.

b. The relation between these two bases can be represented by the following equation：$M_v=M_wU$, where $M_v=\left[\overrightarrow{v_1}\ \overrightarrow{v_2}\right]$, $M_w=\left[\overrightarrow{w_1}\ \overrightarrow{w_2}\right]$, and U is a 2×2 upper triangular matrix. Based on the Gram-Schmidt process performed in part a., find U.【91 交大交研所】

【參考解答】a. $\overrightarrow{w_1}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$, $\overrightarrow{w_2}=\begin{bmatrix}-\frac{1}{10}\\\frac{7}{10}\\\frac{7}{10}\\-\frac{1}{10}\end{bmatrix}$.　b. $U=\begin{bmatrix}2&4\\0&10\end{bmatrix}$.

【習題 2】

Let $T\left(\vec{x}\right)=A\vec{x}$ be a linear transformation from R^2 to R^2 as a counterclockwise rotation through the angle of $90°$ followed by an orthogonal projection on to a line L in R^2, which is spanned by the vector $\begin{bmatrix}-1\\1\end{bmatrix}$.

a. Find the matrix A.

b. If the two steps in part a. are swapped (ie., the orthogonal projection is performed first and followed by the rotation), find the new matrix. 【91 交大交研所】

【參考解答】a. $A=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$. b. $A=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$.

【習題 3】

Assume coordinate of any vector $\vec{x}$ with respect to a basis B is denoted as $\left[\vec{x}\right]_B$.

Consider the basis B_1 of R^2 consisting of vector, $\vec{v_1}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{v_2}=\begin{bmatrix}5\\3\end{bmatrix}$.

(1) If $\left[\vec{x_1}\right]_{B_1}=\begin{bmatrix}2\\1\end{bmatrix}$, find $\vec{x_1}$.

(2) If $\vec{x_2}=\begin{bmatrix}1\\0\end{bmatrix}$, find $\left[\vec{x_2}\right]_{B_1}$.

(3) Consider another basis B_2 of R^2 consisting of vector, $\vec{u_1}=\begin{bmatrix}0\\1\end{bmatrix}$, and $\vec{u_2}=\begin{bmatrix}-1\\0\end{bmatrix}$. If $\left[\vec{x_3}\right]_{B_2}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$, find $\left[\vec{x_3}\right]_{B_1}$. 【92 交大交研所】

【參考解答】(1) $\vec{x_1}=\begin{bmatrix}9\\5\end{bmatrix}$. (2) $\left[\vec{x_2}\right]_{B_1}=\begin{bmatrix}3\\-1\end{bmatrix}$. (3) $\left[\vec{x_3}\right]_{B_1}=\begin{bmatrix}-5c_1-3c_2\\2c_1+c_2\end{bmatrix}$.

【習題 4】

Consider a basis B of R^3 consisting of the following vectors: $\vec{v_1}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$, $\vec{v_2}=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$, $\vec{v_3}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$.

(1) Show that B is an orthonormal basis.

(2) If $\vec{x}=\begin{bmatrix}3\\5\\7\end{bmatrix}$, find its coordinates with respect to the basis B. 【93 交大交研所】

【參考解答】

(1) B is an orthonormal basis.

(2) the coordinates with respect to the basis B is $\begin{bmatrix}5\sqrt{2}\\2\sqrt{2}\\-5\end{bmatrix}$.

■ 方向導數與梯度

【習題 1】

Find the directional derivative of f at P in the direction of $\vec{a}$, where

$f=e^x\cos y$, $P(2,\pi,0)$, $\vec{a}=2\vec{i}+3\vec{j}$. 【中央土研所】

【參考解答】在 $\vec{a}$ 方向的方向導數為 $\nabla f\big|_{(2,\pi,0)}\cdot\frac{\vec{a}}{|\vec{a}|}=-\frac{2e^2}{\sqrt{13}}$

【習題 2】

Given a function $\phi(x,y)=k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$, find the directional derivative of ϕ

along its boundary curve $C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 【95 交大土研所(10%)】

【參考解答】 $\frac{d\phi}{dn}=\frac{2k(y-x)}{\sqrt{a^4y^2+b^4x^2}}$

【習題 3】

Find the directional derivative of $f(x,y)=x^4-3x^3y+x^2y^2$ at $(2,1)$ along the curve $x=t^2+1,\ \ y=t^3$ in the direction of increasing t. 【成大土研所】

【參考解答】$\left.\dfrac{df}{ds}\right|_{P(t=1)}=-\dfrac{48}{\sqrt{13}}$

【習題 4】

已知函數 $F(x,y,z)=axy^2+byz+cz^2x^3$ 在點 $(1,2,-1)$ 處沿著 z 軸的方向有最大的方向導數(directional derivative)，其值爲 64，請問 a , b , c 三個常數值別爲何？【91 中央土研所結構組】

【參考解答】$a=6,\ \ b=24,\ \ c=-8$

■ 散度與旋度

【習題 1】

Given $\vec{R}=x\vec{i}+y\vec{j}+z\vec{k}$, find $\vec{n}$. 【90 台大土研所】

【參考解答】$\nabla\cdot\left(\dfrac{\vec{R}}{\left|\vec{R}\right|^3}\right)=0$

【習題 2】

證明：$\nabla\cdot\left(r^n\vec{r}\right)=(n+3)r^n$ 。【交大土研所】

【參考解答】$\nabla\cdot\left(r^n\vec{r}\right)=(n+3)r^n$

■ 空間曲線的微分幾何

【習題 1】

請利用向量函數(vector function, parametric equation or parametric representation)求 circular cylinder：$x^2+y^2=a^2,\ \ 0\le z\le 2$ 表面($x^2+y^2=a^2$ 所在之面)之單位正交向量。【92 交大土研所甲組】

【參考解答】單位正交向量 $\vec{n}=\dfrac{x\vec{i}+y\vec{j}}{\sqrt{x^2+y^2}}$

【習題 2】

For a temperature distribution $T(x,y,z)=x^2z+yz^2$ in a cone represented by the positive vector as $\vec{r}=u\cos v\vec{i}+u\sin v\vec{j}+2u\vec{k}$, find $\dfrac{dT}{dn}$ at position $P(1,0,2)$ in the outer normal direction $\vec{n}$. 【88 成大土研所丁組】

【參考解答】$\dfrac{dT}{dn}=-\dfrac{7}{\sqrt{5}}$

■ 曲率與扭率

【習題 1】

螺旋線 $\vec{r}(s)=a\cos\dfrac{s}{\omega}\vec{i}+a\sin\dfrac{s}{\omega}\vec{j}+b\dfrac{s}{\omega}\vec{k}$， $\omega=\sqrt{a^2+b^2}$ 求曲率 κ 及扭率 τ 。

【參考解答】$\kappa=\dfrac{a}{a^2+b^2}$， $\tau=\dfrac{b}{a^2+b^2}$

【習題 2】

有一曲線 $x=3\cos t$， $y=\sin t$， $z=4t$,則這一曲線的曲率半徑為何？【93 中央土研所結構組大地組】

【參考解答】曲率半徑 $=\dfrac{25}{3}$

【習題 3】

設 $\vec{r}(t)=3t\cos t\vec{i}+3t\sin t\vec{j}+4t\vec{k}$,求在 $t=0$ 時的 $\vec{e}_t$， $\vec{e}_n$， $\vec{e}_b$ 以及曲率 κ 及扭率 τ 。

【參考解答】$\vec{e}_t=\dfrac{3}{5}\vec{i}+\dfrac{4}{5}\vec{k}$， $\vec{e}_n=\vec{j}$， $\vec{e}_b=-\dfrac{4}{5}\vec{i}+\dfrac{3}{5}\vec{k}$， $\kappa=\dfrac{6}{25}$， $\tau=\dfrac{6}{25}$

■ 向量積分

【習題 1】

請計算$\int_k (x+y)^2 dx-(x^2+y^2)dy$，其中$k$依經過$A(1,1)$，$B(3,2)$，$C(2,5)$爲頂點的三角形圍線。【92 交大運研所】

【參考解答】$\int_k (x+y)^2 dx-(x^2+y^2)dy=-\frac{140}{3}$

【習題 2】

Let the vector field $\vec{F}=\frac{x\vec{i}-z\vec{j}+y\vec{k}}{x^2+y^2+z^2}$, the position vector $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, and the line paths C be on the plane $x=0$ and extend from the point $(0,1,0)$ to the point $(0.-2.0)$. Are the line integrals $\int_C \vec{F}\cdot d\vec{r}$ and $\int_C \vec{F}\times d\vec{r}$ independent of path? Why? Evaluate the line integrals. 【91 台大土研所】

【參考解答】$\int_C \vec{F}\times d\vec{r}=\pi$, $\int_C \vec{F}\times d\vec{r}=-\ln 2\vec{i}$

【習題 3】

設曲面$S: z=2-(x^2+y^2)$，$z\geq 0$,求$\iint_S (x^2+y^2)dA$。

【參考解答】$\iint_S (x^2+y^2)dA=\frac{149\pi}{30}$

【習題 4】

計算$\iiint_V \frac{z}{\sqrt{x^2+y^2+z^2}}dxdydz$，其中$V$爲$a^2\leq x^2+y^2\leq b^2$，$0\leq z\leq\sqrt{x^2+y^2}$所界定之區域。

【參考解答】$\iiint_V \frac{z}{\sqrt{x^2+y^2+z^2}}dxdydz=\frac{(2\sqrt{2}-1)\pi}{3}(b^3-a^3)$

■ 格林定理、高斯散度定理、史托克定理

【習題 1】

Use Green's theorem to evaluate $\oint_C \left(3x^2+y\right)dx+\left(2x+y^3\right)dy$, where C is the circle $x^2+y^2=a^2$. 【93 交大交研所】

【參考解答】$\oint_C \left(3x^2+y\right)dx+\left(2x+y^3\right)dy=\pi a^2$

【習題 2】

求$\oint_C y^2dx+4xydy$，其中C爲拋物線$y=x^2$與直線$y=2x$所圍區域之邊界，且積分路徑採逆時鐘走勢。【91 台科大結構組】

【參考解答】$\oint_C y^2dx+4xydy=\dfrac{64}{15}$

【習題 3】

(1) For a curve $x=t^2+1,\ y=4t-3,\ z=2t^2-6t$, determine the unit tangent vector at the point where $t=2$.

(2) Evaluate $\iint_S \vec{X}\cdot\vec{n}dS=32\pi$, where $\vec{X}=x\vec{i}+y\vec{j}+z\vec{k}$, $\vec{n}$ is the outward unit normal to S, and S is the surface of the sphere $\left(x-1\right)^2+\left(y+3\right)^2+z^2=4$. 【93 成大土研所結構組(20%)】

【參考解答】(1) the unit tangent vector is $\dfrac{2}{3}\vec{i}+\dfrac{2}{3}\vec{j}+\dfrac{2}{3}\vec{k}$, (2) $\iint_S \vec{X}\cdot\vec{n}dS=32\pi$

【習題 4】

A vector field is $\vec{V}=y\vec{i}+x\vec{j}+x^2\vec{k}$, and the surface is described as $S: z=1-\left(x^2+y^2\right),\ 0\le z$, calculate the following flux integral $I=\iint_S \vec{V}\cdot\vec{n}dA$, where $\vec{n}$ is an outer unit normal vector on the surface. 【95 成大土研所乙組(20%)】

【參考解答】$I=\iint_S \vec{V}\cdot\vec{n}dA=\dfrac{\pi}{4}$

【習題 5】

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=y^2\vec{i}+xy\vec{j}+xz\vec{k}$ and $C: x^2+y^2=2ay$, $y=z$. 【台大土研所】

【參考解答】$\oint_C \vec{F}\cdot d\vec{r}=0$

■ 線積分

【習題 1】

Calculate the work done by a force $\overline{F}=X^2\overline{i}-xy\overline{j}$ from point $(1,0)$ to $(-1,0)$ along a curve of $x^2+\dfrac{y^2}{4}=1$ in the upper plane (i.e., $y\geq 0$). 【91 成大土木(15%)】

【參考解答】$w=-\dfrac{10}{3}$

【習題 2】

Compute the line integral $\int_C \overline{f}(r)\cdot d\vec{r}$, where $\overline{F}(r)=y^2\overline{i}-x^2\overline{j}$ C is a straight-line segment from $(0,0)$ to (1.2). 【91 中央化工、材料(10%)】

【參考解答】$\int_C \overline{F}\cdot d\vec{r}=\dfrac{2}{3}$

【習題 3】

求$\overline{F}=(y+z)\overline{i}+(z+x)\overline{j}+(x+y)\overline{k}$沿著曲線 C：$t\overline{i}+t^2\overline{j}+t^3\overline{k}$，0＜ t ＜1 的線積分$\int_C \overline{f}\cdot d\overline{R}=?$ 【91 成大資源(10%)】

【參考解答】$\int_C \overline{F}\cdot d\overline{R}=3$

【習題 4】

Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=x\vec{i}+6\vec{j}+yx\vec{k}$ and C is shown below. 【91 中原化工】

【參考解答】$\int_C \vec{F}\cdot d\vec{r}=-\frac{3}{2}$

【習題 5】

Evaluate $\oint_C zdx+xdy+ydz$, where C is the trace of the cylinder $x^2+y^2=1$ in the plane $y+z=3$. 【91 嘉義機電(30%)】

【參考解答】$\oint_C zdx+xdy+ydz=2\pi$

【習題 6】

If $\vec{F}=\left(3x^2-6yz\right)\vec{i}+\left(2y+3xz\right)\vec{j}+\left(1+4xyz^2\right)\vec{k}$, evaluate line integral $\int_c \vec{F}\cdot d\vec{r}$ along the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(1,1,1)$.
【91 淡江機械(15%)】

【參考解答】$\int_c \vec{g}\cdot d\vec{r}=-6$ ，$\int_c \vec{F}\cdot d\vec{r}=-3$

【習題 7】

Let $\vec{F}=\vec{a}_x 2xy+\vec{a}_y x^2+\vec{a}_z(z-1)$. Evaluate the line integral $\int_{(0,0,0)}^{(1,1,0)}\vec{F}\cdot d\vec{\ell}$ along a parabola $y=x^2$ on the xy plane. 【91 中山機電(10%)】

【參考解答】$\int_{(0,0,0)}^{(1,1,0)}\vec{F}\cdot d\vec{\ell}=1$

【習題 8】

Evaluate the integral $\int_A^B \vec{F}\cdot d\vec{\ell}$, $\vec{F}=2xy\vec{i}+\left(x^2-z^2\right)\vec{j}-3xz^3\vec{k}$, $A(0,0,0)$, $B(2,1,3)$ by performing the integral along

(1) line segment from A to $C(2,1,0)$ to B.

(2) straight line from A to B. 【91 海洋電機固態組(15%)】

【參考解答】(1) $\int_{ABC} \overline{F} \cdot d\vec{\ell} = -50$ (2) $\int_{AB} \overline{F} \cdot d\vec{\ell} = -\frac{79}{2}$

【習題 9】

Let $\overline{F} = -\vec{i} + xyz\vec{j} - y^2\vec{k}$, and let C be given by $x = t$, $y = |t|$, $z = 1$; t: $-1 \to 1$.

Please find $\int_C \overline{F} \cdot d\vec{r} = ?$ 【91 成大製造(8%)】

【參考解答】 $\int_C \overline{F} \cdot d\vec{r} = -\frac{4}{3}$

■ 與路徑無關之線積分

【習題 1】

Let $\overline{F} = \left(yze^{xyz} - 4x\right)\hat{a}_x + \left(xze^{xyz} + z\right)\hat{a}_z$ for all x, y and z.

(1) Verify that F is conservative.

(2) Find a potential function for $\overline{F}$. 【91 台科電機(15%)】

【參考解答】(1)存在ϕ，使得$\nabla\phi = \overline{F}$ ，$\overline{F}$ 爲保守場 (2) $\phi = e^{xyz} - 2x^2 + yz + c$ 爲保守位能

【習題 2】

Find the work done by $\overline{F} = x^2\vec{i} - 2yz\vec{j} + z\vec{k}$ in moving an object along the straight line from $(1,1,1)$ to $(4,4,4)$. 【91 北科化工(15%)】

【參考解答】 $w = -\frac{27}{2}$ ，本題$\overline{F}$ 不是保守場，$\because w = \int \overline{F} \cdot d\vec{r} \neq \int d\phi$

【習題 3】

Evaluate the integral $I = \int_c \left[\left(6xy^2 - y^3\right)dx + \left(6x^2y - 3xy^2\right)dy\right]$ from point $(1,0)$ to point $(3,2)$ along line segment.【91 中興化工(8%)】

【參考解答】$I = 84$

習題 4

Consider the force field $\overline{F} = y^2\bar{i} + 2(xy+z)\bar{j} + 2y\bar{k}$.

(1) Determine the potential function.

(2) Evaluate $\int_{(1,1,1)}^{(2,2,2)} \overline{F}\cdot d\bar{r}$.【91 高科機械(20%)】

【參考解答】$\phi = xy^2 + 2yz + c$，$\int_{(1,1,1)}^{(2,2,2)} \overline{F}\cdot d\bar{r} = 13$

習題 5

空間有一力場$\overline{F}(x,y,z) = y\bar{i} + z\bar{j} + x\bar{k}$，求$\vec{F}$沿曲線$C$所作的功$W = \oint_C \overline{F}\cdot d\bar{r} = ?$

其中封閉曲線C由右式定義$x+y=2$，$x^2+y^2+z^2 = 2(x+y)$(本題請忽略C的方向，只求$|W|$)

【參考解答】$\int_C \overline{F}\cdot d\bar{r} = 2\sqrt{2}\pi$

習題 6

Evaluate $\int \overline{F}\cdot d\overline{R}$, where $\overline{F} = zy\sin(xy)\bar{i} + zx\sin(xy)\bar{j} + (2e^z - \cos(xy))\bar{k}$, and $\overline{R}$ is the position vector along the curve C from $(1,1,2)$ to $(1,-1,6)$.【90 北科光電(10%)】

【參考解答】$\int \overline{F}\cdot d\overline{R} = -4\cos 1 + 2(e^6 - e^2)$

習題 7

Let C be a path on the paraboloid $x^2+y^2-z=0$ from the initial point $(1,0,1)$ to the terminal point $(0,1,1)$; otherwise, C is arbitrary.

(1) What is the value of the line integral $I = \int_C \frac{-ydx + xdy + zdz}{x^2+y^2}$ along the path C?

Is it independent of path?

(2) Why? (Prove your answer in(1).)【89 台大土木(19%)】

【參考解答】(1) $\int_C \frac{-ydx+xdy+zdz}{x^2+y^2}=\left[\tan^{-1}\frac{y}{x}+z\right]_{(1,0,1)}^{(0,1,1)}$ (2)當路徑爲 C_1，$I=\frac{\pi}{2}$。

當路徑爲 C_2，$I=-\frac{3}{2}\pi$

■ 向量面積分

習題 1

$I=\iint_S \frac{xy}{z}dA$，其中 S 爲 $z=x^2+y^2$ 對應於第一象限之 $4\le x^2+y^2\le 9$ 之部分。【90 淡江環工(25%)】

【參考解答】$I=\frac{1}{24}\left[37^{\frac{3}{2}}-17^{\frac{3}{2}}\right]$

習題 2

$\vec{v}=y\vec{i}-z\vec{j}+yz\vec{k}$, find the surface integral $I=\iint_s \vec{v}\cdot\vec{n}dA$ for s: $x=\sqrt{y^2+z^2}$, $y^2+z^2\le 1$.【91 成大土木(15%)】

【參考解答】$I=0$

習題 3

一向量場方程式爲 $\overline{F}=[x,y,z]$。一曲面的方程式 S ：$\vec{r}=\left[u\cos v,u\sin v,u^2\right]$，$0\le u\le 4$，$-\pi\le v\le\pi$ 。問通過此曲面的向量通量爲何？【91 中興環工(10%)】

【參考解答】flux $I=128\pi$

習題 4

Integrate the surface integral $\iint\limits_S \overline{F}\cdot\vec{n}dA$, where $\overline{F}=\left[e^y,0,ze^x\right]$, $\vec{n}$: units normal vector of S, S: $\vec{r}=[u,2u,v]$, $-1\le u\le 1$, $0\le v\le 3$.【91 中興材料(20%)】

【參考解答】$y=2x$，$-1\le x\le 1$，$0\le z\le 3$，投影到 xz 面處理 $\vec{n}dA=\dfrac{\nabla\phi}{\left|\nabla\phi\cdot\vec{j}\right|}dxdz$，

$\phi=2x-y$，$\nabla\phi=\left(2\vec{i}-\vec{j}\right)dxdz$，$\iint_S \overline{F}\cdot\vec{n}dA=3\left(e^2-e^{-2}\right)$。

習題 5

Evaluate the surface integral $\iint \overline{F}\cdot\vec{n}dA$, where $\overline{F}=\left(y^3,x^3,z^3\right)$. Surface S:

$x^2+4y^2=1$, $x\ge 0$, $y\ge 0$, $0\le z\le h$.【91 成大水利(15%)】

【參考解答】$\iint_S \overline{F}\cdot\vec{n}dA=\dfrac{17}{64}h$

習題 6

Calculate the flux of water through the parabolic cylinder S: $y=x^2$,

$0\le x\le 2, 0\le z\le 3$, if the velocity vector is $\overline{F}=y\vec{i}+2\vec{j}+xz\vec{k}$, speed being

measured in $m^3/\sec$.【90 中興化工(10%)】

【參考解答】$\iint_S \overline{F}\cdot\vec{n}dA=12$

習題 7

Evaluate $\iint_S zds$, with S the part of the plane $x+y+z=6$ lying above the

rectangle $0\le x\le 3, 0\le y\le 2$.【91 北科車輛(20%)】

【參考解答】$\iint_S zds=21\sqrt{3}$

習題 8

對某一函數 $f(x,y,z)=y$，試求此函數在一平滑表面(smooth surface) $z=x^2$，
$0\le x\le 2$，$0\le y\le 3$ 上之面積積分(surface integral)。【90 屏科環工(15%)】【91 台科電子(5%)】

【參考解答】$I=\dfrac{9\sqrt{17}}{2}+\dfrac{9}{8}\sinh^{-1}4$

習題 9

Find the area of the following surface $z = x^2 + y^2,\ 0 \le z \le 10$.【89 成大造船(17%)】

【參考解答】$A = \frac{\pi}{6}\left[41^{\frac{3}{2}} - 1\right]$

習題 10

Evaluate $\iint_S \vec{A} \cdot \vec{n} dS$, where $\vec{A} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ where is located in the first quadrant.【89 中興機械(15%)】

【參考解答】$\iint_S \vec{A} \cdot \vec{n} dS = 24$

習題 11

If $\vec{F} = x\vec{i} + y\vec{j}$ Calculate the surface integral $\int \vec{F} \cdot (\vec{n} dA)$ over the part of the surface $z = 4 - x^2 - y^2$ that is above the (X, Y) plane.【91 淡江物理(15%)】

【參考解答】$\iint_S \vec{F} \cdot \vec{n} dA = 16\pi$

■ 平面 Green's 定理

習題 1

已知一力場爲 $\vec{F} = \left(y - \sin(x) e^x\right)\vec{i} + \left(\cos 2y - x\right)\vec{j}$，

(1) 求 $\vec{F}$ 沿路徑 C_1 所作的功。

(2) 利用格林(Green)定理及(1)之結果，計算 $\vec{F}$ 沿路徑 C_2 所作的功。【90 台科營建(15%)】

【參考解答】(1) $w_1 = \frac{1}{2}\sin 2 - 1$ (2) $\int_{C_2} \vec{F} \cdot d\vec{r} = -\pi - 3 - \frac{1}{2}\sin 2$

習題 2

Verify Green's theorem by $\vec{F} = 3y\vec{i} - 2xy\vec{j}$ along circle $(x-3)^2 + (y-2)^2 = 16$.【89 成大土木(15%)】

【參考解答】$\oint_C 3ydx - 2xydy = \iint_R \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(3y)\right]dxdy = -112\pi$

習題 3

Evaluate the integral $\oint_C x^2dx + xy^2dy$. Where C: $C_1 + C_2$ as indicated in Fig 1.

【89 交大環工(15%)】

【參考解答】$\oint_C \left(x^2dx + xy^2dy\right) = -\frac{\pi}{4} + \frac{16}{3}$

習題 4

(1) Evaluate $\oint_C (xy + y^2)dx + x^2dy$ where C is the boundary of the region bounded by $y = x$ and $y = x^2$.

(2) Verify your answer by Green's theorem.【91 逢甲電機、電子(15%)】

【參考解答】(1) C_1：$y = x^2$，C_2：$y = x$，$C = C_1 + C_2$，而 $\int_C \left(xy + y^2\right)dx + x^2dy = -\frac{1}{20}$

(2) $\oint_C \left(xy + y^2\right)dx + x^2dy = -\frac{1}{20} = \iint_R \left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}\left(xy + y^2\right)dxdy\right]$ 成立

習題 5

Evaluate the line integral $\oint y^2dx - x^2dy$ along the curve C, which is the union of C_1, C_2, and C_3. These curves are described by C_1: $y = 0$, x from 0 to 2; C_2: $x = 2$, y from 0 to 4; C_3: $y = x^2$, x from 2 to 0, and verify Green's theorem.【90 台大化工(20%)】

【參考解答】

on C_1 ： $y = 0$ ， $x = 0 \rightarrow x = 2$ ， $\int_{C_1} y^2dx - x^2dy = 0$

on C_2 ： $x=2$ ， $y=0\to 4$ ， $\int_{C_2} y^2dx-x^2dy=-16$

on C_3 ： $y=x^2$ ， $dy=2xdx$ ， $x=2\to x=0$ ， $\int_{C_3} y^2dx-x^2dy=\frac{8}{5}$ ，

$\int_C \left(y^2dx-x^2dy\right)=-\frac{72}{5}$

習題 6

(1) State Green's theorem.

(2) Prove that $A=\frac{1}{2}\oint_C (xdy-ydx)$.

(3) Prove that $A=-\oint_C tdx=\oint xdy$.

(4) Use(2)or(3)to calculate the area of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.【90 雲科營建(10%)】

【參考解答】(1) $\oint_C fdx+gdy=\iint_R\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy$ (2) $\frac{1}{2}\oint_C\left(xdy-ysx\right)=A$

(3) $\oint_C xdy=A$ (4) $A=\pi ab$

習題 7

Please find the area of the region bounded by the x-axis and one arch of the cycloid (擺線) given by: $x=a(t-\sin t)$ and $y=a(1-\cos t)$.【91 中山環工(15%)】

【參考解答】on C_1 ： $x=a\left(t-\sin t\right)$ ， $y=a\left(1-\cos t\right)$ ， $dy=a\sin tdt$ ， $\int_{C_1} xdy=3\pi a^2$

on C_2 ： $x=0\to 2\pi a$ ， $y=0$ ， $dy=0$ ， $\int_{C_2} xdy=0$ ， $A=3\pi a^2$

習題 8

$\overline{F}=x^2y\vec{i}-xy^2\vec{j}$ find

(1) $\oint_C \overline{F}\cdot d\vec{r}$, c is the contour from the origin to $(1,1)$ to $(2,0)$ and then back to the origin.

(2) $\iint_S \left(\nabla\times\overline{F}\right)\cdot d\vec{s}$.

(3)Verify Stokes' theorem.【90 海洋電機、電波組(13%)】

【參考解答】(1)on C_1：$y=x$，$d\vec{r}=(\vec{i}+\vec{j})dx$，$\overline{F}=x^3\vec{i}-x^3\vec{j}$，$\int_{C_1}\overline{F}\cdot d\vec{r}=0$

on C_2：$y=-x+2$，$d\vec{r}=(\vec{i}-\vec{j})dx$，$F=x^2(2-x)\vec{i}-x(2-x)^2\vec{j}$，$\int_{C_2}\overline{F}\cdot d\vec{r}=\frac{4}{3}$

on C_3：$y=0$，$x=2\to x=0$，$d\vec{r}=dx\underline{i}$，$\overline{F}=0$，$\int_{C_3}\overline{F}\cdot d\vec{r}=0$，得$\oint_C\overline{F}\cdot d\vec{r}=\frac{4}{3}$

(2)$\iint_S(\nabla\times\overline{F})\cdot\vec{n}dA=\frac{4}{3}$

習題 9

Let $\phi(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ be a scalar field and $\vec{F}(x,y)=\frac{-y}{x^2+y^2}\vec{i}+\frac{x}{x^2+y^2}\vec{j}$ a vector field defined in (x,y) plane.

(1) Evaluate the directional derivative of ϕ at the point $(1,1)$ in the direction

$\vec{n}=\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}$

(2) Evaluate $\nabla\times(\nabla\phi)$.

(3) Evaluate the line integrals $\oint_C\overline{F}\cdot d\vec{r}$ over the two different closed curves C_1 and C_2 as indicated in the following figures.【90 台大機械(10%)】

【參考解答】(1)$\left.\frac{d\phi}{ds}\right|_{\vec{n}}=0$ (2)$\nabla\times\nabla\phi=\nabla\times\overline{F}$ (3)$\oint_{C_1}\overline{F}\cdot d\vec{r}=0$，$\oint_{C_2}\overline{F}\cdot d\vec{r}=2\pi$

習題 10

Assume the vector $\overline{F}=(-y\vec{i}+x\vec{j})/(x^2+y^2)$, evaluate the contour integral

$\oint_C\overline{F}\cdot d\vec{s}=$? Where the contour C is along

(1)A unit circle $x^2+y^2=1$.

(2)A unit circle $(x-2)^2+y^2=1$.

(3)An ellipse $5x^2+6y^2=14$

(4)Make a comment on your answer in (1), (2) and (3).【90 中興機械(20%)】

【參考解答】(1)C：$x^2+y^2=1$，$\oint_C\overline{F}\cdot d\vec{r}=2$ (2)C：$(x-2)^2+y^2=1$，$\oint_C\overline{F}\cdot d\vec{r}=0$

(3) C ：$5x^2+6y^2=14$ ，$\oint_C \vec{F}\cdot d\vec{r}=2\pi$ (4)當(1)與(3)中，$(0,0)$在C內，$\vec{F}$在C內不爲保守場繞C一圈，$\tan^{-1}\frac{y}{x}$增加2π ，當(2)中，$(0,0)$在C外，$\vec{F}$在C內爲保守場繞C一圈，作功量爲零，or $\tan^{-1}\frac{y}{x}$沒變

習題 11

Consider the vector, $\vec{F}(t)=\frac{2x}{x^2+y^2}\vec{e}_x+\frac{2y}{x^2+y^2}\vec{e}_y$

(1) If $\vec{F}$ is a force field, is this force conservative? If yes, please find the corresponding potential function.

(2) Find $\oint_C \vec{F}\cdot d\vec{r}, C$ is any closed path not passing through $(0,0)$.【90 中原物理 (20%)】

【參考解答】(1)存在ϕ ，使得$\nabla\phi=\vec{F}$

$$\frac{\partial\phi}{\partial x}=\frac{2x}{x^2+y^2} \text{，} \phi=\ln\left(x^2+y^2\right)+h(y)$$

$$\frac{\partial\phi}{\partial y}=\frac{2y}{x^2+y^2} \text{，} \phi=\ln\left(x^2+y^2\right)+g(x)$$

上下二式相比，$h(y)=g(x)=c$，$\phi=\ln\left(x^2+y^2\right)+c$當$(0,0)$在封閉曲線$c$外，則$\vec{F}$在$c$內爲保守場；當$(0,0)$在$c$內，則$\vec{F}$在封閉曲線$c$內不爲保守場

(2)當$(0,0)$在c外，$\oint_C \vec{F}\cdot d\vec{r}=0$ ；當$(0,0)$在c內$\oint_C \vec{F}\cdot d\vec{r}=0$

習題 12

Find $\oint_C \vec{F}\cdot d\vec{r}$ for any closed curve c not passing through point $(0,0)$

$\vec{F}=\left(\frac{y}{x^2+y^2}+x^2\right)\vec{i}+\left(\frac{-x}{x^2+y^2}-2y\right)\vec{j}$.【91 北科土木(25%)】

【參考解答】$\oint_C \vec{F}\cdot d\vec{r}=-2\pi$

習題 13

Let $\overline{F}(x,y)=\frac{1}{(x+y)^2}\vec{i}+\frac{1}{(x+y)^2}\vec{j}$ be a vector function and let $\phi(x,y)$ be a scalar function such that $\nabla\phi=\overline{F}$.

(1) Find $\phi(x,y)$. Is $\overline{F}(x,y)$ a conservative field on the entire (x,y) plane excluding the origin?

(2) Evaluate the integral $\int_C \overline{F}\cdot d\vec{r}$ along a line segment C: $x-y=1$ starting from point $(0,-1)$ to point $(1,0)$.

(3) For this particular problem, does the closed loop integral $\oint_C \overline{F}\cdot d\vec{r}$ along any closed curve C enclosing origin equal zero or not?【91 台大機械(15%)】

【參考解答】(1) $\phi=-\frac{1}{x+y}+c$ 直線 $x+y=0$ 上每一點都爲二階極點，即奇異點。當封閉曲線 C 內有奇異點，$\overline{F}$ 在 C 內不爲保守場，當封閉曲線 C 內無奇異點，$\overline{F}$ 在 C 內爲保守場。

(2) $\int_C \overline{F}\cdot d\vec{r}=\infty$ Note $\int_{-1}^{1}\frac{1}{t^2}dt\neq-\frac{1}{t}\Big|_{-1}^{1}=-2$，積分路徑上二階點 $t=0$，積分值發散，$\int_{(0,-1)}^{(1,0)}\overline{F}\cdot d\vec{r}=-2$

(3) $\oint_C \overline{F}\cdot d\vec{r}$ 發散，∵路徑上有二階極點

習題 14

(1) What kind of vector is called conservative.

(2) Explain that is $\overline{F}$ is a conservative vector field, then its line integral is independent of oath.

(3) By (2), explain that the circulation of $\overline{F}$ is zero.【91 北科冷凍(18%)】

【參考解答】(1)當 $\nabla\times\overline{F}=0$，存在 $\phi=\phi(x,y,z)$，使得 $\nabla=\overline{F}$，若 $\overline{F}$ 在區域 R 內

沒有奇異點，$\overline{F}$ 在區域 R 內爲保守場。

(2)當 $\overline{F}$ 爲保守場 $\int_a^b \overline{F}\cdot d\bar{r}=\phi(b)-\phi(a)$ 積分值與路徑無關，只和端點之 ϕ 值有關。

(3) $\oint_C \overline{F}\cdot d\bar{r}=0$

習題 15

Evaluate $\oint_C \dfrac{ydx-xdy}{x^2+y^2}$ where $C: x^2+y^2=1$.【90 交大環工(15%)】

【參考解答】on c ： $x=\cos\theta$ ， $y=\sin\theta$ ， $dx=-\sin\theta d\theta$ ， $dy=\cos\theta d\theta$ ，

$\int_C \dfrac{ydx-xdy}{x^2+y^2}\int_0^{2\pi}(-\sin\theta\cdot\sin\theta-\cos\theta\cdot\cos\theta)d\theta=-2\pi$

習題 16

Calculate $\int_C \dfrac{-y^3dx+xy^2dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2+y^2=4$. Can we apply the Green's theorem to calculate the integration, why?【90 中原土木(15%)】

【參考解答】 $\int_C \dfrac{-y^3dx+xy^2dy}{(x^2+y^2)^2}=\pi$

習題 17

A force field is described by $\overline{F}=-\dfrac{y}{(x^2+y^2)}\bar{i}+\dfrac{x}{(x^2+y^2)}\bar{j}$.

(1) Express $\overline{F}$ in circular cylindrical coordinate.
(2) Is $\overline{F}$ a conservative force field?
(3) Calculate the work done by $\overline{F}$ in encircling the unit circle (centering at the origin) once counterclockwise.【91 台師大機電(15%)】

【參考解答】(1) $\overline{F}=\dfrac{1}{r}\bar{e}_\theta$ (2) $\nabla\times\overline{F}=\dfrac{1}{r}\begin{vmatrix}\bar{e}_r & r\bar{e}_\theta & \bar{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ 0 & 1 & 0\end{vmatrix}=0$，存在 ϕ 使得 $\nabla\phi=\overline{F}$，當 $(0,0)$

在封閉曲線 c 內，具有奇異點 $(0,0)$，$\overline{F}$ 不爲保守場，當 $(0,0)$ 在封閉曲線 c 外，$\overline{F}$ 在 c 內爲保守場。 (3) $\oint_C \overline{F}\cdot d\vec{r}=2\pi$

習題 18

Consider a vector field $\overline{F}=\frac{x}{x^2+y^2}\vec{i}+\frac{y}{x^2+y^2}\vec{j}$ defined in (x,y) plane. Let C denote a closed circle of radius 1 centered at the origin, D be the region bounded by C, and $\vec{n}$ the unit vector normal to the circle.

(1) Evaluate $\nabla\cdot\overline{F}$.

(2) Evaluate the line integral $\oint_C \overline{F}\cdot\vec{n}ds$ along the closed circle C.

(3) Does the Divergence Theorem $\oint_C \overline{F}\cdot nds=\oiint_D \nabla\cdot\overline{F}dA$ hold true in this case? Is not, please give the reason why the theorem does not apply here.

【參考解答】(1) $\nabla\cdot\overline{F}=0$ (2)on c，$\vec{n}ds=\vec{r}d\theta$，$\oint_C \overline{F}\cdot\vec{n}ds=2\pi$

(3) $\oint_C \overline{F}\cdot\vec{n}ds\neq\iint_S(\nabla\cdot\overline{F})dA$，$\because$ 在 c 內有奇異點 $(0,0)$ 散度定理不成立

習題 19

Let the vector field $\overline{F}(x,y,z)=\frac{x\vec{i}-z\vec{j}+y\vec{k}}{x^2+y^2+z^2}$, the position vector $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, and the line paths C be on the plane $x=0$ and extend from the point $(0,-2,0)$.

Are the line integrals $\int_C \overline{F}\cdot d\vec{r}$ and $\int_C \overline{F}\times d\vec{r}$ independent of path? Why?

Evaluate the line integrals.【91 台大土木(18%)】

【參考解答】當 C 在上半平面 $\int_C \overline{F}\cdot d\vec{r}=\pi$，當 C 在下半平面 $\int_C \overline{F}\cdot d\vec{r}=-\pi$，積分值與路徑有關，$\int_C \overline{F}\times d\vec{r}=-\ln 2\vec{i}$ 積分值與路徑無關，$\because$ 沒有帶入路徑參數即可積出

習題 20

Evaluate the following integrals $\oint_C \frac{\partial w}{\partial n} ds$ where, $w = 3x^2 y - y^3 + y^2$, c: $25x^2 + y^2 = 25$.【87 成大土木(20%)】

【參考解答】$\oint_C \frac{\partial w}{\partial n} ds = 10\pi$

習題 21

(1) Find the gradient $\nabla\phi$ of the function $\phi = \ln r$, $r = \sqrt{x^2 + y^2}$, if in the xy plane

$C_1 = \left[(x,y)\middle|(x-2)^2 + y^2 = 1\right]$, $D_1 = \left[(x,y)\middle|(x-2)^2 + y^2 < 1\right]$

$C_2 = \left[(x,y)\middle|(x^2 + y^2 = 1)\right]$, $D_2 = \left[(x,y)\middle|x^2 + y^2 < 1\right]$

$C_3 = \left[(x,y)\middle|(x-1)^2 + y^2 = 1\right]$, $D_3 = \left[(x,y)\middle|(x-1)^2 + y^2 < 1\right]$

$\vec{n} =$ the outward unit normal vector on C_i with respect to D_i, $\vec{r} = x\vec{i} + y\vec{j}$ find the positively-oriented contour integral.

(2) $\oint_{C_1} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds$

(3) $\oint_{C_2} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds$

(4) $\oint_{C_3} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds$【87 台大土木(20%)】

【參考解答】(1) $\phi = \ln r$ ，$\nabla\phi = \nabla \ln r = \frac{1}{r}\nabla r = \frac{\vec{r}}{r^2}$ ，$\nabla\phi = \frac{1}{x^2+y^2}\left[x\vec{i} + y\vec{j}\right]$

(2) $\oint_{C_1} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds = 0$ (3) $\oint_{C_2} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds = 1$ (4) $\oint_{C_3} \frac{\vec{r}}{2\pi r^2} \cdot \vec{n} ds = \frac{1}{2}$

■ GAUSS 散度定理

習題 1

Let D be the hemisphere bounded by $x^2 + y^2 + (z-1)^2 = 9$, $1 \le z \le 4$, and the plane $z = 1$, find the flux over this hemisphere when $\vec{F} = x\vec{i} + y\vec{j} + (z-1)\vec{k}$.【90 台科

電子(15%)】

【參考解答】$\iint_D \vec{F}\cdot\vec{n}dA = 54\pi$

習題 2

Please verify the Gauss theorem for the volume shown on the figure for the vector

$\vec{v} = x\vec{e}_x + y\vec{e}_y + 2z\vec{e}_z$.【89 台大化工(15%)】

【參考解答】$\oiint_S \vec{v}\cdot\vec{n}dA = \frac{2}{3}$，故 $\oiint_S \vec{v}\cdot\vec{n}dA = \iiint_v (\nabla\cdot\vec{v})dV$ 得證

習題 3

Evaluate $\int_S \vec{F}\cdot d\vec{S}$, where vector $\vec{F}(x,y) = xy^2\vec{i} + x^2y\vec{j} + y\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 2$, $-1 < z < 2$, and $x^2 + y^2 \le 2$ when $z = \pm 2$.【89 清大工程科學(15%)】

【參考解答】$\int_S \vec{F}\cdot\vec{n}dS = 8\pi$

習題 4

Verify divergence theorem for the vector field $\vec{F}(x,y,z) = \vec{a}_x x + \vec{a}_y 2x + \vec{a}_z xyz$ over a cube bounded by $x = 0$, $x = 1, y = 0$, $y = 1, z = 0$ and $z = 1$.

(1) Compute $\nabla\cdot\vec{F}(x,y,z)$.

(2) Perform the volume integral.

(3) Perform the surface integral.【91 中山通訊(15%)】

【參考解答】(1) $\nabla\cdot\vec{F} = 1 + xy$ (2) $\iiint \nabla\cdot\vec{F}dv = \frac{5}{4}$

(3) $\iint_S \vec{F}\cdot\vec{n}dA = \iint_{左} + \iint_{右} + \iint_{上} + \iint_{下} + \iint_{前} + \iint_{後} \vec{F}\cdot\vec{n}dA$，$\oiint \vec{F}\cdot\vec{n}dA = \frac{5}{4}$

習題 5

Let $\sum$ be the closed surface consisting of the surface $\sum_1$ of the cone

$z^2 = x^2 + y^2$ for $0 \le x^2 + y^2 \le 1$ and the flat cap $\sum_2$ consisting of the disk $x^2 + y^2 \le 1$, $z = 1$, as shown in the following figure. Illustrate Gauss's Divergence Theorem by separately computing both sides of the equation for a vector field $\vec{F}(x,y,z) = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$.【90 台科電機(10%)】

【參考解答】$\iint_{\sum_1} \vec{F}\cdot\vec{n}dA + \iint_{\sum_2} \vec{F}\cdot\vec{n}dA = \iiint_R \nabla\cdot\vec{F}dv = \pi$

習題 6

Shown that a region T with surface S has the volume $V = \iint_S zdxdy = \frac{1}{3}\iint_S zdxdy + xdydz + ydzdx$.【91 中興土木(10%)】

【參考解答】

$V = \frac{1}{3}\iint_S zdxdy + xdydz + ydzdx = \frac{1}{3}\iint_S \left(x\vec{i} + y\vec{j} + z\vec{k}\right)\cdot\vec{n}dA = \frac{1}{3}\iiint_S \nabla\cdot\left(x\vec{i} + y\vec{j} + z\vec{k}\right)dv$

$= \frac{1}{3}\iiint_R 3dv$

習題 7

(1)Evaluate $\iint_S \left(x\hat{a}_x + y\hat{a}_y + 3z\hat{a}_z\right)\cdot d\vec{A}$ over S：$x^2 + y^2 + z^2 = 4$ and

(2)Verify the divergence theorem with (1). Here x, y, z are Cartesian coordinates and $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are unit vectors along x, y and z axes, respectively.【91 交大電信(15%)】

【參考解答】(1) $\iint_S \left(x\vec{i} + y\vec{j} + 3z\vec{k}\right)\cdot dA = \frac{160}{3}\pi$ (2) $\iint \left(x\vec{i} + y\vec{j} + 3z\vec{k}\right)\cdot dA = \frac{160}{3}\pi$，散度定理成立

習題 8

Let $T(x,y,z) = x^2 + y^2 + z^2$ represent temperature and let the flow of heat be given by the vector field $\vec{F} = -\vec{\nabla}T$. Find the flux of heat out of the sphere $x^2 + y^2 + z^2 = 4$.【91 台大化工(10%)】

【參考解答】$\iint_S \vec{F}\cdot\vec{n}dA = -64\pi$

習題 9

Find the (1) volume and (2) centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 6$, $z = 0$ assuming the density to be a constant. The region R is shown in the Figure as follow.【91 北科機電(15%)】

【參考解答】$\bar{z} = \frac{8}{5}$，$\bar{y} = 3$，$\bar{x} = \frac{3}{4}$

習題 10

(1) Prove that for any closed, the surface integral $\oiint_{\infty} \vec{n} ds$ is equal to zero vector of surface.

(2) A cone with surface function $z^2 = 2(x^2 + y^2)$ in $0 \le z \le 2\sqrt{2}$ is shown below.

Evaluate $S = \iint_{S_\infty} \vec{n} ds = ?$ S_∞: the wall surface of the cone.【91 中原土木(10%)】

【參考解答】(1) $\oiint \vec{n} ds = 0$ (2) $\vec{S} = -4\pi \vec{k}$

習題 11

設 S 唯一封閉曲線，R 為其所包圍之區域，試證明

$$\oiint_S \frac{\vec{r}}{r^3} \cdot \vec{n} dA = \begin{cases} 0 & \text{原點在S外} \\ 4\pi & \text{原點在S內 (本例又稱為Gauss Theorem)} \\ 2\pi & \text{原點在S上} \end{cases}$$

【參考解答】(1)若原點不在 S 內，則 $\frac{\vec{r}}{r^3}$ 在 S 內，具有連續一階偏導數，

$\therefore \oiint_S \frac{\vec{r}}{r^3} \cdot \vec{n} dA = 0$ (2)若原點在 S 內，則在原點附近，挖一半徑 $\in$ 之圓，S^*：

$x^2 + y^2 + z^2 = \in^2$ 則在 S^* 與 S 所包圍之區域內 R^*，$\frac{\vec{r}}{r^3}$ 具有連續一階偏導數，

$\oiint_S \frac{\vec{r}}{r^3} \cdot \vec{n} dA = 4\pi$ (3)設原點在 S 表面，定義積分主值，則 $\oiint_S \frac{\vec{r}}{r^3} \cdot \vec{n} dA = 2\pi$

習題 12

(1) Evaluate $\iint_S \left(7x\vec{i} - z\vec{k}\right)\cdot\vec{n}dA$ over the sphere $S:\ x^2+y^2+z^2=4$ by integration directly, where $\vec{n}$ is the outer unit normal vector of S.

(2) Repeat (1) by using the divergence theorem.【90 淡江電機(20%)】

【參考解答】(1) $\iint_S \left(7x\vec{i} - z\vec{k}\right)\cdot\vec{n}dA = 64\pi$

(2) $\iint_S \left(7x\vec{i} - z\vec{k}\right)\cdot\vec{n}dA = \iiint (7-1)\,dv = 6v = 6\cdot\frac{4}{3}\pi\cdot 2^3 = 64\pi$

習題 13

Find $I = \oiint_S \left[3y^2 zdxdy + e^x dydz - ye^x dxdz\right]$ over the circular cylinder S : $x^2+y^2\le 1,\ |z|\le 2$ including top, bottom and cylinder.【90 中興土木(10%)】

【參考解答】令 $\overline{F} = e^x\vec{i} - ye^x\vec{j} + 3y^2z\vec{k}$ ，$\nabla\cdot\overline{F} = 3y^2$ ，$I = 3\pi$

習題 14

$\overline{F} = x^3\vec{a}_x + x^2y\vec{a}_y + x^2z\vec{a}_z$ Verify the divergence theorem when the region is bounded by a cylinder $x^2+y^2=16$ and the planes at $z=0$ and $z=2$.【87 元智電機(20%)】

【參考解答】$\iint_{柱}\overline{F}\cdot\vec{n}dA = 512\pi$ ，$\iint_{上}\overline{F}\cdot\vec{n}dA = 128\pi$ ，$\iint_{下}\overline{F}\cdot\vec{n}dA = 0$ ，

$\oiint_S \overline{F}\cdot\vec{n}dA = 640\pi = \iiint \nabla\cdot\overline{F}dv$ 成立

習題 15

由散度定理，證明曲線座標中之散度

$$\nabla\cdot\overline{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(F_1h_2h_3\right)}{\partial u_1} + \frac{\partial\left(F_2h_1h_3\right)}{\partial u_2} + \frac{\partial\left(F_3h_2h_1\right)}{\partial u_3}\right]$$

【參考解答】$\nabla\cdot\overline{F} = \lim_{\nabla v\to 0}\frac{1}{\Delta v}\oiint \overline{F}\cdot\vec{n}dA$ ，

$$\oiint \vec{F}\cdot\vec{n}dA=\iint_{左}\vec{F}\cdot\vec{n}dA+\iint_{右}\vec{F}\cdot\vec{n}dA+\iint_{上}\vec{F}\cdot\vec{n}dA+\iint_{下}\vec{F}\cdot\vec{n}dA+\iint_{前}\vec{F}\cdot\vec{n}dA+\iint_{後}\vec{F}\cdot\vec{n}dA$$

$$=\Delta u_1\Delta u_2\Delta u_3\left[\frac{\partial(F_1h_2h_3)}{\partial u_1}+\frac{\partial(F_2h_1h_3)}{\partial u_2}+\frac{\partial(F_3h_1h_2)}{\partial u_3}\right]$$

，$\nabla\cdot\vec{F}=\lim_{\Delta V\to 0}\frac{1}{\Delta v}\oiint\vec{F}\cdot\vec{n}dA=\frac{1}{h_1h_2h_3}\left[\frac{\partial(F_1h_2h_3)}{\partial u_1}+\frac{\partial(F_2h_1h_3)}{\partial u_2}+\frac{\partial(F_3h_1h_2)}{\partial u_3}\right]$

習題 16

Let $\vec{v}=rz\hat{e}_r+3\hat{e}_\theta+rz^2\hat{e}_z$, evaluate the surface integral $\int_\zeta \hat{n}\cdot\vec{n}da$, including the top, bottom, and side, for a cylinder $0\le r\le 3$，$0\le z\le 6$.【89 交大機械(17%)】

【參考解答】$\iint\vec{v}\cdot\vec{n}dA=\int_0^6\int_0^{2\pi}\int_{R=0}^{3}(2z+2zr)rdrd\theta dz=972\pi$

習題 17

(1) 已知一穩定流流體之速度向量$\vec{v}=-y^2\vec{i}+2\vec{j}$，證明該流體具不可壓縮性並試求流體內任一質點之運動路徑方程式。

(2) 向量場$F=2z^2-y^2-x^2$通過一矩形體在$0\le x\le 1$，$0\le y\le 2$，$0\le z\le 4$表面S：證明F為一諧和函數(harmonic function)及$\iint_S\frac{\partial F}{\partial n}dA=0$。【91 中山海運(20%)】

【參考解答】(1)$\nabla\cdot\vec{v}=0$，為不可壓縮，令運動路徑上任一點位置$\vec{r}=x\vec{i}+y\vec{j}$，

$d\vec{r}\,//\,\vec{v}$，$\frac{dx}{-y^2}=\frac{dy}{2}$，$dx=-\frac{1}{2}y^2dy$，$x=-\frac{1}{6}y^3+c$

(2)$\nabla^2F=\frac{\partial^2F}{\partial x^2}+\frac{\partial^2F}{\partial y^2}+\frac{\partial^2F}{\partial z^2}=-2-2+4=0$，$F$ is a harmonic function，

$\iint\frac{\partial F}{\partial n}dA=\iint_S\nabla F\cdot\vec{n}dA=\iiint\nabla\cdot\nabla Fdv=\iiint\nabla^2Fdv=\iiint 0dv=0$得證

■ Stoke 氏定理

習題 1

Verify Stoke's theorem for the vector field $\overline{F}(x,y,z)=\vec{a}_x x+\vec{a}_y x+\vec{a}_z 2xy$ using the hemisphere $x^2+y^2+z^2=4$, $z<0$.

(1) Compute $\nabla\times\overline{F}(x,y,z)$.

(2) Perform the line integral.

(3) Perform the surface integral.【90 中山通訊(15%)】

【參考解答】(1) $\nabla\times\overline{F}=2x\vec{i}-2y\vec{j}+\vec{k}$ (2)

$\oint_C \overline{F}\cdot d\vec{r}=\int_{2\pi}^{0}\left(-4\cos\theta\sin\theta+4\cos^2\theta\right)d\theta=-4\pi$

(3) $\iint_S\left(\nabla\times\overline{F}\right)\cdot\vec{n}dA=\int_0^{2\pi}\int_0^2\frac{2r^2\cos 2\theta}{\sqrt{4-r^2}}rdrd\theta-4\pi=-4\pi$

習題 2

Verify Stoke's theorem for $\overline{f}=(2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$, C is its boundary.【91 中興機械(10%)】

【參考解答】 $\int_C\overline{f}\cdot d\vec{r}=\int_0^{2\pi}\left(-2\sin\theta\cos\theta+\sin^2\theta\right)d\theta=\pi$ 故

$\int_C\overline{f}\cdot d\vec{r}=\iint_S\left(\Delta\times\overline{f}\right)\cdot\vec{n}dA=\pi$

習題 3

Given $\overline{V}=4y\hat{i}+x\hat{j}+2z\hat{k}$, find $\int\left(\nabla\times\overline{V}\right)\cdot\left(\hat{n}dA\right)$ over the hemisphere $x^2+y^2+z^2=16$, $z\geq 0$.【91 淡江物理(15%)】

【參考解答】 $\int_S\left(\nabla\times\overline{V}\right)\cdot\vec{n}dA=\oint_C\overline{F}\cdot d\vec{r}=\int_0^{2\pi}\left(-64\sin^2\theta+16\cos^2\theta\right)d\theta=-4\pi$

習題 4

Verify the Stoke's theorem for the case where the vector field $v=xz\hat{j}$, and where S is the surface $z=4-y^2$, cut off by the planes $x=0$, $z=0$ and $y=x$.【90 清大材料(10%)】

【參考解答】on C_1：$x=0$，$z=4-y^2$，$y=2\to y=0$，$d\vec{r}=\left(\vec{j}-2y\vec{k}\right)dy$，$\vec{v}=0$，

$\int_{C_1}\vec{v}\cdot d\vec{r}=0$

on C_2：$x=y$，$z=4-y^2$，$y=0\to y=2$，$d\vec{r}=\left(\vec{i}+\vec{j}-2y\vec{k}\right)dy$，

$\vec{v}=y\left(4-y^2\right)\vec{j}$，$\int_{C_2}\vec{v}\cdot d\vec{r}=4$

on C_3：$z=0$，$x=2\to x=0$，$y=2$，$d\vec{r}=dx\vec{i}$，$\vec{v}=0$，$\int_{C_3}\vec{v}\cdot d\vec{r}=0$

$\iint\left(\nabla\times\vec{v}\right)\cdot\vec{n}dA=4=\int_C\vec{v}\cdot d\vec{r}$

習題 5

兩維空間的向量函數$\overline{f}\left(x,y\right)$定義如下：$f_x=-y$，$f_y=x$。$A$、$B$、$C$、$D$四點的座標爲：$A\left(0,0\right)$、$B\left(0,1\right)$、$C\left(1,1\right)$、$D\left(1,0\right)$。

(1) 沿路徑$A\to B\to C$，計算$\int\overline{f}\cdot d\bar{r}$的值。

(2) 沿路徑$A\to D\to C$，計算$\int\overline{f}\cdot d\bar{r}$的值。

(3) 從A到C，沿路徑$y=x^2$，再計算$\int\overline{f}\cdot d\bar{r}$之值

(4) 直接計算$\int\overline{f}\cdot d\bar{r}$沿$A\to B\to C\to D$繞一封閉矩形的積分值，並用比例解釋 Stoke’s 定理。【91 成大光電(20%)】

【參考解答】(1)on AB，$x=0$，$y=0\to y=1$，$dx=0$ $\int\overline{f}\cdot d\bar{r}=0$

on BC，$x=0\to 1$，$y=1$，$dy=0$，$\int\overline{f}\cdot d\bar{r}=-1$

(2)on AD，$x=0\to 1$，$y=0$，$dy=0$，$\int\overline{f}\cdot d\bar{r}=0$

on DC，$x=1$，$y=0\to 1$，$dx=0$，$\int\overline{f}\cdot d\bar{r}=1$

(3)on AC，$y=x^2$，$x=0\to 1$，$\int_C\overline{f}\cdot d\bar{r}=\frac{1}{3}$

(4) $\oint_C \vec{f}\cdot d\vec{r} = -\iint\left(\nabla\times\vec{f}\right)\cdot\vec{k}dA = -2$

習題 6

利用 Stoke theorem，求下列線積分 $I = \oint_C y^2dx + z^2dy + x^2dz$ ，C 爲連續 $(1,0,0)$ ，$(0,1,0)$ ，$(0,0,1)$ 之三角形。【91 海洋船研通訊組(20%)】

【參考解答】$I = \iint_S\left(\nabla\times\vec{f}\right)\cdot\vec{n}dA = \iint_S -2dxdy = -1$

習題 7

Given $\vec{F} = xy\vec{i} + \left(x^2+y\right)\vec{j} + xy^2z\vec{k}$ and C is a closed curve prescribed in the figure as coarse curve shown below. Calculate the line integral $\oint_C \vec{F}\cdot d\vec{r}$ and C. Hint: Use Stoke's Theorem.【90 中原機械(20%)】

【參考解答】$\oint_C \vec{F}\cdot d\vec{r} = \int_{y=0}^{2}\left(-8y^4 + 2y^6 + \frac{1}{2}y^2\right)dy = -\frac{1396}{105}$

習題 8

Let $\vec{F}(x,y,z)$ be a vector field and $\vec{a}$ be an arbitrary unit vector. It can be shown that $\vec{a}\cdot\left(\nabla\times\vec{F}\right) = \lim_{\Delta A\to 0}\frac{1}{\Delta A}\oint_C \vec{F}\cdot d\vec{r}$ where the loop C enclosing the area ΔA is perpendicular to $\vec{a}$. From this, explain the physical meaning of $\nabla\times\vec{F}$ in terms of its magnitude and direction.【87 中山電機(10%)】

【參考解答】$\nabla\times\vec{F}$ 表單位面積旋轉量，$\left(\nabla\times\vec{F}\right)\cdot\vec{n} = \lim_{\Delta A\to 0}\frac{1}{\Delta A}\oint_C \vec{F}\cdot d\vec{r}$，表 $\nabla\times\vec{F}$ 在 $\vec{n}$ 方向上之單位面積旋轉量

習題 9

請在柱座標系統下，求取以下之積分 $\oint_C \vec{V}\cdot d\vec{R} = ?$ 其中

$\overline{V}=-r^2\cos\theta\vec{e}_r+r^2\vec{e}_\theta+r\sin\theta\vec{e}_z$，$C=C_1\cup C_2\cup C_3\cup C_4\cup C_5$ 而圖中 $S=S_1\cup S_2$ 爲 C 所圍成之曲面，S_1 爲在 $z=3$ 之半徑爲 2 之 $\frac{1}{4}$ 圓，S_2 爲 $\frac{1}{4}$ 圓筒面，C_2 爲 $\frac{1}{4}$ 半徑爲 25 支圓弧。【89 中央土木(15%)】

【參考解答】$\oint_C \overline{V}\cdot d\overline{R}=-6-4\pi+\frac{8}{3}=\frac{-10}{3}-4\pi$

習題 10

Given that $\overline{F}=x^2\hat{a}_x-xz\hat{a}_y-y^2\hat{a}_z$, calculate the circulation of $\overline{F}$ around the (closed) path shown in the following figure.【88 台科電機(10%)】

【參考解答】$\iint_{1,5,4}\left(\nabla\times\overline{F}\right)\cdot\vec{n}dA=\iint zdxdy=\frac{1}{6}$，$\oint_C \overline{F}\cdot d\vec{r}=-\frac{1}{6}$

習題 11

Let S be the part of the cylinder $z=1-x^2$ for $0\le x\le 1$, $-1\le y\le 2$ verify stoke's theorem if $\vec{F}=xy\hat{i}+yz\hat{j}+xz\hat{k}$.【89 台科電子(12%)】

【參考解答】$\oint_C \overline{\mathrm{F}}\cdot d\vec{r}=\int_{C_1}\overline{F}\cdot d\vec{r}+\int_{C_2}\overline{F}\cdot d\vec{r}+\int_{C_3}\overline{F}\cdot d\vec{r}+\int_{C_4}\overline{F}\cdot d\vec{r}=-3$，

$\iint_S\left(\nabla\times\overline{F}\right)\cdot\vec{n}dA=\int_{-1}^{2}\int_0^1(-2xy-x)dxdy=-3$，故 $\iint_S\left(\nabla\times\overline{F}\right)\cdot\vec{n}dA=\oint_C\overline{F}\cdot d\vec{r}$

■ 向量內積與外積

習題 1

Find the unit vector that is orthogonal to both

$\vec{u}=\vec{i}-4\vec{j}+\vec{k}$ *and* $\vec{\mathrm{v}}=2\vec{i}+3\vec{j}$.【91 中山環工(15%)】

【參考解答】$\phi=\frac{1}{\sqrt{134}}\left(-3\vec{i}+2\vec{j}+11\vec{k}\right)$ 爲所求

習題 2

Use Gram-Schmidt process to find three orthonomal vectors from
$V_1 = [1\ 7\ 1\ 7]$，$V_2 = [0\ 7\ 2\ 7]$，$V_3 = [1\ 8\ 16]$【91 台科電子(10%)】

【參考解答】

$u_1 = \frac{1}{10}[0\ 1\ 0\ -1]$, $u_2 = \frac{1}{\sqrt{2}}[-1\ 0\ 1\ 0]$, $u_3 = \frac{1}{\sqrt{2}}[0\ 1\ 0\ -1]$, $\{u_1, u_2, u_3\}$ is an orthonormal set

習題 3

(1) Give a vector $\vec{a} = \hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3$, where $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are the 3 unit vectors along the x-, y-, and z-axis. Find the equation for a plane normal to this vector and passing the origin of the coordinate system.

(2) Find the equation for the plane normal to this vector and passing the point (0,1,0). 【91 中央光電(10%)】

【參考解答】(1) $x + 2y + 3z = 0$ (2) $x + 2y + 3z = 2$

習題 4

A parallelogram has two incident sides extending from $(0, 1, -2)$ to $(1, 2, 2)$ and from $(0, 1, -2)$ to $(1, 4, 1)$. Find the area of this parallelogram.【90 台科電機(10%)】

【參考解答】平行四邊形面積$= \sqrt{85}$

習題 5

Find the point (x, y, z) on the given plane $x - y + 2z = 4$, that is closest to the point A$(2, 0, -1)$, and the shortest distance.【90 中原化工(10%)】

【參考解答】(1) $P\left(\frac{8}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ (2) $d = \frac{4}{\sqrt{6}}$

習題 6

Through $(-1, 2, 3)$ perpendicular to both the lines $x = -1 + 3t$, $y = 2$, $z = 3 - t$ and $x = -1 - t$, $y = 2 + 3t$, $z = 3 + t$. 【90 北科自動化(10%)】

【參考解答】$x = 3t - 1$，$y = -2t + 2$，$z = 9t + 3$

習題 7

求一含$(1,2,1)$，$(-1,1,3)$，$(-2,-2,-2)$之平面。【90 北科環境(10%)】

【參考解答】$11x-12y+5z=-8$

習題 8

求空間中二直線間最短距離d：

L_1：$\dfrac{x-2}{3}=\dfrac{y-5}{2}=\dfrac{z-1}{-1}$

L_2：$\dfrac{x-4}{-4}=\dfrac{y-5}{4}=\dfrac{z+2}{1}$　【85 中央土木 10%】

【參考解答】$d=\dfrac{48}{\sqrt{437}}$

習題 9

Find the distance between the two straight lines: $(x,y,z)=(3+t,1-2t,2+2t)$ and $(x,y,z)=(7+t,1-2t,-3+2t)$, where t is a parameter.【90 清大動機(10%)】

【參考解答】$d=\sqrt{37}$

習題 10

Let $\vec{a}$ and $\vec{b}$ be nonzero vectors, prove that the vector $\vec{c}=|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ bisects the angle between $\vec{a}$ and $\vec{b}$.【90 海洋電機通訊組(15%)】

【參考解答】$\alpha=\beta$

習題 11

Find an unit projection vector $\vec{u}$ of line $x=2t+1$, $y=3$, $z=-t+2$ on the plane $3x+y-2z+6=0$【89 中興土木(20%)】

【參考解答】$u=\dfrac{1}{\sqrt{21}}\left(2\vec{i}-4\vec{j}+\vec{k}\right)$

習題 12

Given Point $A(1,0,2)$, $B(4,5,0)$, $C(0,-3,5)$, $D(-2,0,7)$, and $E(0,-1,7)$ in the space, please find:

(1) The normal unit vector of plane CDE.

(2) The equation, which is the perpendicular bisector of line AB.

(3) The coordinate of piercing point M, where the line AB intersects the plane CDE.【91 清大動機(15%)】

【參考解答】

(1) $\vec{n}=\frac{1}{3}\left(\vec{i}+2\vec{j}-2\vec{k}\right)$ (2) $3x+5y-2z=18$ 為所求 (3) $F\left(-\frac{22}{17},-\frac{65}{17},\frac{60}{17}\right)$ 為所求

習題 13

Assume that there are two lines whose parametric descriptions are, respectively, $(1,-2,1)t+(2,4,5)$ and $(2,4,4)r+(2,0,4)$. Will these two lines intersect? Explain. (If your answer is TES, find the point of intersection.)【91 高科通訊(10%)】

【參考解答】L_1 與 L_2 有交點，交點座標 $x=3$ ， $y=2$ ， $z=6$

習題 14

(1) The points $A(1,-2,1)$, $B(0,1,6)$ and $C(-3,4,-2)$, form a triangle. Find the angle between the line AB and the line from A to the midpoint of the line BC.

(2) Find the equation of a plane passing through $(-6,1,1)$ and perpendicular to $-2\mathrm{i}+4\vec{j}+\vec{k}$.【91 交大機械(10%)】

【參考解答】(1) $\theta=\cos^{-1}\left(\sqrt{\frac{2}{35\times55}}21\right)$ (2) $-2x+4y+z=17$ 為所求

■ 純量三重積與向量三重積

習題 1

An unknown vector $\vec{x}$ satisfies the relations: $\vec{x}\cdot\vec{b}=\beta$, and $\vec{x}\times\vec{b}=\vec{c}$. Try to express $\vec{x}$ in terms of β, $\vec{b}$, and $\vec{c}$.【91 清大物理(10%)】

【參考解答】$k_2 = \frac{1}{\vec{b}\cdot\vec{b}}$，$\vec{x} = \frac{1}{\vec{b}\cdot\vec{b}}\left[\beta\vec{b} + \vec{b}\times\vec{c}\right]$

習題 2

以向量的概念，計算出 a、b、c、d 四點所圍出的四面體的體積，此四點的座標分別是：

$a = (0,1,2)$，$b = (5,5,6)$，$c = (1,2,1)$，$d = (3,3,1)$【91 中央化工、化工(10%)】

【參考解答】$V = \frac{7}{6}$

習題 3

One corner of a rectangular parallelepiped is at $(-1,2,2)$ and three incident sides extend from this point to $(0,1,1)$, $(-4,6,8)$, and $(-3,-2,4)$. Find the volume of this parallelepiped.【90 台科電機(10%)】

【參考解答】$V = 18$

習題 4

Simplify $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$.【91 中央物理(5%)】

【參考解答】$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

習題 5

若 $\vec{a}$、$\vec{b}$、$\vec{c}$、$\vec{d} \in R^3$，且 4 向量共面，求 $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$。【89 北科電機(10%)】

【參考解答】$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0$

習題 6

(1) If the vector $\vec{F},\vec{G},\vec{H}$ are linear independent in R^3 space and the $\vec{V}$ also a vector in R^3 space, then please prove

$\vec{V}=\dfrac{[\vec{V}\vec{G}\vec{H}]}{[\vec{F}\vec{G}\vec{H}]}\vec{F}+\dfrac{[\vec{V}\vec{H}\vec{F}]}{[\vec{F}\vec{G}\vec{H}]}\vec{G}+\dfrac{[\vec{V}\vec{F}\vec{G}]}{[\vec{F}\vec{G}\vec{H}]}\vec{H}$ Where [ ] is the triple vector product.

(2) If the vector $\vec{F}=(1,\ 2,\ 3)$, $\vec{G}=(2,\ 4,\ 2)$, $\vec{H}=(2,\ 1,\ 3)$ and $\vec{V}=(11,\ 13,\ 16)$,

Please find the $\vec{V}=\alpha\vec{F}+\beta\vec{G}+\gamma\vec{H}$, $\alpha=?$ $\beta=?$ $\gamma=?$ By using the (1) results. 【90 北科電機(20%)】

【參考解答】(1) $\vec{V}=\dfrac{[\vec{V}\vec{G}\vec{H}]}{[\vec{F}\vec{G}\vec{H}]}\vec{F}+\dfrac{[\vec{V}\vec{H}\vec{F}]}{[\vec{F}\vec{G}\vec{H}]}\vec{G}+\dfrac{[\vec{V}\vec{F}\vec{G}]}{[\vec{F}\vec{G}\vec{H}]}\vec{H}$ (2) $\alpha=1$，$\beta=2$，$\gamma=3$，

$\vec{V}=\vec{F}+2\vec{G}+3\vec{H}$

習題 7

Find the volume of a tetrahedron(四面體) bounded by coordinate surfaces ($x=0$, $y=0$, $z=0$) and the plane $x+\dfrac{y}{2}+\dfrac{z}{3}=1$.【90 中山材料(15%)】

【參考解答】$V=1$

習題 8

For a given basis in a three-dimensional space $g_1=[1,-1,2]$, $g_2=[0,1,1]$, $f_3=[-1,-2,1]$

(1) Find the coordinate of the vector $h=[3,3,6]$ in this basis.

(2) Let us introduce a set of reciprocal base vectors g^1, g^2, g^3 so that $g_i\cdot g^i=1$, $i=1,2,3$, $g^j=0$, $i\neq j$. Find the reciprocal base vectors g^1, g^2, g^3.

(3) Find the coordinate of the vector $h=[3,3,6]$ in the reciprocal basis.【87 成大土木(20%)】

【參考解答】

(1) $a=2$，$b=3$，$c=-1$

(2) $g^1=\dfrac{1}{6}(3\vec{i}-\vec{j}+\vec{k})$，$g^2=\dfrac{1}{6}(-3\vec{i}+3\vec{j}+3\vec{k})$，$g^3=\dfrac{1}{6}(-3\vec{i}-\vec{j}+\vec{k})$

(3) $\alpha=12$，$\beta=9$，$\gamma=-3$，$h=12g^1+9g^2-3g^3$

習題 9

(1) Prove Cauchy-Schwarz inequality $\left|\vec{F}\cdot\vec{G}\right| \le \left\|\vec{F}\right\|\left\|\vec{G}\right\|$, where $\vec{F}$ and $\vec{G}$ are vectors.

(2) Use Cauchy-Schwarz inequality to verify $\left\|\vec{F}+\vec{G}\right\| \le \left\|\vec{F}\right\|+\left\|\vec{G}\right\|$.【88 台科電機(10%)】

【參考解答】

(1) $\left|\vec{F}\cdot\vec{G}\right| = \left\|\vec{F}\right|\cdot\left|\vec{G}\right|\cdot\cos\theta\right| \le \left|\vec{F}\right|\cdot\left|\vec{G}\right|$ 得證 (2) $\left|\vec{F}+\vec{G}\right|^2 \le \left(\left|\vec{F}+\vec{G}\right|\right)^2$ 故 $\left|\vec{F}+\vec{G}\right| \le \left|\vec{F}\right|+\left|\vec{G}\right|$

■ 向量微分

習題 1

Find the volume of the solid bounded below the paraboloid $z = 4 - x^2 - y^2$ and above by the plane $z = 4 - 2x$.【91 交大機械(5%)】

【參考解答】$V = \dfrac{\pi}{2}$

習題 2

A funnel, as shown in the figure, whose angle at the outlet is 60° and whose outlet has a cross-sectional area of $0.5\ cm^2$, contains water. At time $t = 0$ the outlet is opened and the water flows out. Determine the time when the funnel will be empty, assuming that the initial height of water is $h(0) = 10$ cm. The velocity with witch a liquid issues from an orifice is $v = 0.6(2gh)^{1/2}$.【89 中央機械(25%)】

【參考解答】when empty, $h = 0$, $t = 99.6$

習題 3

The position vector in a cylindrical coordinates (R,θ,Z) is given by $\vec{r} = R\hat{e}_R + Z\hat{e}_Z$. Show what the velocity and Newton's 2^{nd} Law of motion, $\vec{V} = \dfrac{d\vec{r}}{dt}$ and $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$, for a particle of mass m can be written in component from in cylindrical coordinates as $(V_R, V_\theta, V_Z) = (\dot{R}, R\dot{\theta}, \dot{Z})$ and

$\left(F_R, F_\theta, F_Z\right)=\left[m\left(\ddot{R}-R\dot{\theta}^2\right), m\left(R\ddot{\theta}+2\dot{R}\dot{\theta}\right), m\ddot{Z}\right]$.【89 中興精密(20%)】

【參考解答】$\vec{F}=m\left[\left(\ddot{R}-R\dot{\theta}^2\right),\left(\mathrm{R}\ddot{\theta}+2\dot{\mathrm{R}}\ddot{\theta}\right),\ddot{z}\right]$

習題 4

A bead moves on a disk toward the edge, the position vector being $\vec{r}(t)=t^2\vec{b}$ where $\vec{b}$ denotes a unit vector in radial direction, rotating together with the disk with constant angular speed ω in the counter clock wise sense. Find the acceleration of the bead. You have to use vector differential calculus to derive your result. No score if you directly employ a mechanic formula.【87 交大機械(20%)】

【參考解答】$\vec{v}=2t\vec{e}_b+t^2w\vec{e}_\theta$, $\vec{a}=\left(2-t^2w^2\right)\vec{e}_b+4tw\vec{e}_\theta$

習題 5

Find the stream line of the vector field $\vec{F}=-y^2\vec{i}+2\vec{j}+\vec{k}$ through the point $(2,0,4)$.

【91 中山、海洋環工(10%)】

【參考解答】代入點$(2,0,4)$，$c_1=4$，$c_2=-8$，$2x+\frac{1}{3}y^3=4$，$y-2z=-8$ stream line 爲二曲面交線

習題 6

The path of motion is given by a vector function $\vec{r}(t)=t\vec{i}-t^2\vec{j}$. Find the corresponding tangential and normal acceleration.【91 交大土木(15%)】

【參考解答】$\vec{e}_t=\frac{1}{\sqrt{1+4t^2}}\left(\vec{i}-2t\vec{j}\right)$，$\vec{a}_t=\frac{4t}{1+4t^2}\left(\vec{i}-2t\vec{j}\right)$，$\vec{a}_n=\frac{1}{1+4t^2}\left[-4t\vec{i}-2\vec{j}\right]$

習題 7

A helix is described by $\vec{r}(t)=\vec{a}_x 2\cos\left(\frac{t}{2}\right)+\vec{a}_y 2\sin\left(\frac{t}{2}\right)+\vec{a}_z t$.

(1)Write $d\vec{r}(t)=\vec{A}(t)dt$. Determine the vector $\vec{A}(t)$.

(2)Find a unit tangent vector to the curve at $(2,0,0)$.

(3)Find the length of the curve from $(2,0,0)$ to $(0,2,\pi)$.【90 中山通訊(15%)】

【參考解答】

(1) $\vec{A}=-\sin\frac{t}{2}\vec{i}+\cos\frac{t}{2}\vec{j}+\vec{k}$ ， $x=2\cos\frac{t}{2}$ ， $y=2\sin\frac{t}{2}$ ， $z=t$ (2) $\vec{e}=\frac{1}{\sqrt{2}}\left(\vec{j}+\vec{k}\right)$

(3) $s=\sqrt{2}\pi$

習題 8

Find the total length of four-cusped hypocycloid $\vec{r}(t)=a\cos^3 t\vec{i}+a\sin^3 t\vec{j}$.【90 彰師電機(10%)】

【參考解答】 $s=6a$

習題 9

Let s be the path $s(t)=\left(2t,t^2,\ln t\right)$, defined for $t>0$. Find the area length of s between the points $(2,1,0)$ and $(4,4,\ln 2)$.【90 清大工程科學(10%)】

【參考解答】 $s=3+\ln 2$

習題 10

一質點於 x-y 平面之位置向量(position vector) $\vec{F}(t)$ 是由

$\vec{F}(t)=p(t)\vec{i}+p'(t)\vec{j}$，$t\geq 0$ 所描述，其中 $p(t)$ 之控制方程式爲 $p''(t)+2p^3(t)=0$；

$p(0)=1$， $p'(0)=0$ 。

(1) 試繪出此質點於 x-y 平面之運動軌跡圖。

(2) 求此質點之最小運動速率，並於 x-y 平面標示其發生點。【90 台科營建(15%)】

【參考解答】

(1) $x=p(t)$ ， $y=p'(t)$ ， $y^2+x^4=1$ 爲質點運動軌跡方程式

(2)最小速率 $v=\left(\frac{316}{54}\right)^{\frac{1}{2}}$ ，發生點 $\left(\frac{1}{6},\pm\frac{1}{36}\sqrt{1295}\right)$

習題 11

若一心臟線之極座標表示爲$r = 2(1+\cos\theta)$，求此心臟線所包絡之面積？曲線長？【91 成大資源(10%)】

【參考解答】$A=6\pi$ ，曲線長$s=16$

習題 12

If a particle is attracted toward the origin by a force whose magnitude is proportional to the distance r of the particle from the origin, how much work is done when the particle is moved from the point (0, 1) to the point (1, 2) along the path $y=1+x^2$, assuming a coefficient of friction μ between the particle and the path?【91 台師大光電(20%)】

【參考解答】$W=2+\frac{2}{3}\mu$

習題 13

空間中一曲線以下式定義 $\vec{f}(t)=e^t(\cos 2t)\vec{i}+e^t(\sin 2t)\vec{j}+e^t\vec{k}$

(1) 請以弧長(arc length) S 表示此一曲線之參數式(以$t=0$點爲參考點)並請說明此一轉換是正確的。

(2) 在$t=\pi/4$ ，請找曲線之

a.單位切向量$\vec{T}$　　　　b.單位法向量$\vec{n}$

c.雙法線向量(binormal)$\vec{B}$　　d.曲率K【90 中央土木(20%)】

【參考解答】

(1)$\vec{f}(t)=\left(\frac{s}{\sqrt{6}}+1\right)\left[\cos\left(2\cdot\ln\left(\frac{s}{\sqrt{6}}+1\right)\right)\vec{i}+\sin\left(2\ln\left(\frac{s}{\sqrt{6}}+1\right)\right)\vec{j}+\vec{k}\right]$

(2)$\vec{T}=\frac{1}{\sqrt{6}}\left(-2\vec{i}+\vec{j}+\vec{k}\right)$ ，$\vec{n}=\frac{-1}{\sqrt{5}}\left(\vec{i}+2\vec{j}\right)$ ，$k=\frac{\sqrt{5}}{3}e^{-\frac{\pi}{4}}$ ，$\vec{B}=\frac{1}{\sqrt{30}}\left(2\vec{i}-\vec{j}+5\vec{k}\right)$

習題 14

Given the position vector $\vec{r}=t\vec{i}+t^2\vec{j}+\frac{2t^3}{3}\vec{k}$, where $\vec{i}$, $\vec{j}$, $\vec{k}$ are rectangular unit vectors.

(1) Find the curvature k, where $k = \left|\frac{d\vec{T}}{ds}\right|$, $\vec{T} = \frac{d\vec{r}}{ds}$.

(2) Find the unit vector $\vec{B}$, where $\vec{B} = \vec{T} \times \vec{N}$ and $\vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds}$.【91 交大機械(25%)】

【參考解答】(1) $k = \frac{2}{\left(1+2t^2\right)^2}$ (2) $\vec{B} = \frac{1}{1+2t^2}\left[2t^2\vec{i} - 2t\vec{j} + \vec{k}\right]$

習題 15

For the circular helix curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$，

$\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k}$，$t \geq 0$, find

(1)are length s, and in terms of s the (2)unit tangent vector $\vec{e}_t$
(3)unit normal vector $\vec{e}_n$ (4)unit binormal vector $\vec{e}_b$
(5)curvature k and torsion τ

【參考解答】

(1) $s = \sqrt{a^2+b^2}\,t$

(2) $\vec{e}_t = -\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}\vec{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}\vec{j} + \frac{b}{\sqrt{a^2+b^2}}\vec{k}$

(3) $\vec{e}_n = -\cos\frac{s}{\sqrt{a^2+b^2}}\vec{i} - \sin\frac{s}{\sqrt{a^2+b^2}}\vec{j}$

(4) $\vec{e}_b = \left(b\sin\frac{s}{\sqrt{a^2+b^2}}\vec{i} - b\cos\frac{s}{\sqrt{a^2+b^2}}\vec{j} + a\vec{k}\right)\frac{1}{\sqrt{a^2+b^2}}$

(5) $\tau = \frac{a^2 b}{\left(a^2+b^2\right)^3}$

習題 16

A position vector $\vec{P}(t)$ is given, $\vec{P}(t) = [\cos t + t\sin t]\vec{i} + [\sin t - t\cos t]\vec{j} + t^2\vec{k}$, $t >$ 0 determine the normal component of the acceleration, the curvature, and the unit normal vector.【91 淡江環工(20%)】

【參考解答】$\vec{e}_n$ is the unit normal vector, $\vec{a}_n$ is normal component of acceleration

習題 17

Evaluate the area of a sphere using

(1) $A(s)=\iint_S dA=\iint \left|r_u \times r_v\right| dudv$ and

(2) $A(s)=\iint \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$.【91 彰師機電(30%)】

【參考解答】(1) $A=4\pi r^2$ (2) $A=4\pi r^2$

習題 18

There is a new museum on Taoyuan city. The surface of the dome on the new museum is given by $\vec{r}(u,\ v)=20\sin u\cdot\cos v\vec{i}+20\sin u\cdot\sin v\vec{j}+20\cos u\vec{k}$ where $0\le u\le\frac{\pi}{3}$, $0\le v\le 2\pi$ and $\vec{r}$ is in meters. Find the surface area of the dome.【91 元智機械(15%)】

【參考解答】$A=400\pi$

習題 19

The area element of a surface $\vec{r}$ is $dA=\left|\overline{N}\right|dudv$, where $\overline{N}$ is the normal vector of the surface. Find the area of a torus surface

$\vec{r}(u,v)=(a+b\cos v)\cos u\vec{i}+(a+b\cos v)\sin u\vec{j}+b\sin v\vec{k}$.【91 彰師機械(15%)】

【參考解答】$A=4\pi^2 ab$

習題 20

(1) Evaluate the line integral $\int_{(0,2)}^{(1,\pi)} \sin xy(ydx+xdy)$.

(2) Evaluate the double integral $\iint_R f(x,y)dxdy$, where $f(x,y)=\cos(x^2+y^2)$, R :

$x^2+y^2 \le \frac{\pi}{2}$, $x \ge 0$.【89 清大電子(20%)】

【參考解答】(1) $\int_{(0,2)}^{(1,\pi)} \sin xy\left(ydx+xdy\right)=2$ (2) $\iint_R \cos\left(x^2+y^2\right)dxdy=\frac{\pi}{2}$

習題 21

Find $\int_0^{\frac{1}{2}}\int_0^{1-2y} e^{\frac{x}{x+2y}}dxdy$.【87 台大生物環境(15%)】

【參考解答】$\int_0^{\frac{1}{2}}\int_0^{1-2y} e^{\frac{x}{x+2y}}dxdy=\frac{1}{4}(e-1)$

習題 22

Find $\iint_R \left(x^2+y^2\right)dxdy=\frac{8}{3}$.【90 朝陽資訊(15%)】

【參考解答】$\iint_R \left(x^2+y^2\right)dxdy=\frac{8}{3}$

習題 23

Find the flux of $\vec{F}=x\vec{i}+y\vec{j}+z\vec{k}$ across the part of the sphere $x^2+y^2+z^2=4$ lying between the planes $z=1$ and $z=2$.【91 北科化工(20%)】【89 台科電機(10%)】

【參考解答】$I=8\pi$

習題 24

Find the surface area of the portion of the sphere $x^2+y^2+z^2=a^2$ that is above the xy-plane and within the cylinder $x^2+y^2=b^2$ where $0<b<a$.【91 成大工程科學(15%)】

【參考解答】$A=2\pi a^2\left[1-\frac{\sqrt{a^2-b^2}}{a}\right]$

■ 方向導數與梯度

習題 1

For a temperature distribution $T(x,y,z)=x^2z+yz^2$, in a cone represented by the position vector as $\vec{r}=u\cos v\vec{i}+u\sin v\vec{j}+2u\vec{k}$, find $\dfrac{dT}{dn}$ at position $P(1,0,2)$ in the outer normal direction $\vec{n}$.【88 成大土木(20%)】

【參考解答】$\dfrac{dT}{dn}=\dfrac{7}{\sqrt{5}}$

習題 2

已知函數 $F(x,y,z)=axy^2+byz+cz^2x^3$在點$(1,2,-1)$處沿著 z 軸的方向有最大的方向導數(directional derivative)，其值為 64，請問a，b，c三個常數值分別為何？【91 中央土木(9%)】

【參考解答】$a=\pm 6$ ，$b=\pm 24$ ，$c=\pm 8$

習題 3

Let the electric potential (i.e. the voltage) be given by $V(x,y,z)=3x^2y-xz$. If a positive charge is placed at $P(1,1,-1)$, in what direction will the charge begin to move? (Note: It is known, from electric field theory, that such a charge will begin to move in the direction of maximum rate of voltage drop.)【91 雲科機械(25%)】

【參考解答】$-\nabla V=-\left(7\vec{i}+3\vec{j}-\vec{k}\right)$即為質點運動方向

習題 4

The temperature distribution in a homogeneous spherical solid filling the closed region $x^2+y^2+z^2\le 1$ and time t is given by $u=\left(z^2-z\right)e^{-2t}$. Let $\vec{n}$ be the unit outer normal on the boundary of the sphere. Find the point at which $\partial u/\partial n$ is minimum.【91 成大土木(15%)】

【參考解答】

$\dfrac{\partial u}{\partial n}$之極小值為$-\dfrac{1}{8}e^{-2t}$ ，發生於$\cos\theta=\dfrac{1}{4}$即$z=\dfrac{1}{4}$ ，$x^2+y^2=\dfrac{15}{16}$上

習題 5

Given a analytic function $f(z)=F_1(x,y)+iF_2(x,y)$, where $z=x+iy$ and $i=\sqrt{-1}$.

If the real part $F_1(x,y)$ and the imaginary part $F_2(x,y)$ of $f(z)$ serve as the components of a vector $\vec{F}$, i.e. $\vec{F}=F_1\vec{i}+F_2\vec{j}$ where $\vec{i}$ and $\vec{j}$ denote the unit vector in $x-$ and $y-$ direction respectively. Then, is the vector $\vec{F}$ a conservative one? Why?【91 成大土木(10%)】

【參考解答】$\frac{\partial F_1}{\partial x}=\frac{\partial F_2}{\partial y}$ ，$\frac{\partial F_2}{\partial x}=-\frac{\partial F_1}{\partial y}$ ，$\vec{F}$ is not a conservative

習題 6

We would like to evaluate the directional derivative of a scalar field $V(x,y,z)=xy+x+z+1$ at the origin.

(1) At first let us find the derivative along the direction $(1,1,1)$. Choose a nearby point $\Delta\vec{r}=(\Delta t,\Delta t,\Delta t)$.

a. Find ΔV, the increment of V from the origin to $\Delta\vec{r}$.

b. Determine the derivative $\lim_{|\Delta\vec{r}|\to 0}\frac{\Delta V}{\Delta\vec{r}}$.

(2)Consider the derivative along any direction in the $xy-$plane. We should now use $\Delta\vec{r}=(\Delta x,\Delta y,0)$. Which direction will give the maximum derivative?【91 中山通訊(15%)】

【參考解答】

(1) $\Delta V=2dt$ ，$\lim_{|\Delta\vec{r}|\to 0}\frac{\Delta V}{\Delta\vec{r}}=\frac{2}{\sqrt{3}}$ (2)當 $\Delta y=0$ ，即 $\Delta\vec{r}=\Delta x\vec{i}$ 方向導數 $\frac{\Delta V}{|\Delta\vec{r}|}=-1$ 爲最大

習題 7

Determine the maximum and minimum rate of change of the function $\phi(x,y,z)=xyz$ at the point $(1,1,1,)$.【91 海洋電機(15%)】

【參考解答】maximum rate of change $|\nabla\phi|=\sqrt{3}$ ，minimum rate of change $-|\nabla\phi|=-\sqrt{3}$

習題 8

Construct the tangential plane passing through an arbitrary point $P(x_0, y_0, z_0)$ on an ellipsoidal surface given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.【91 清大物理(10%)】

【參考解答】得 $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$ 爲切平面方程式

習題 9

(1) Suppose $[x,y,z]=x\vec{i}+y\vec{j}+z\vec{k}$ denotes a vector function, where x,y,z are Cartesian coordinates. If we have a function $f(x,y,z)=2x^2+3y^2+z^2$, find its directional derivative at the point $P:(2,1,3)$ in the direction of the vector $\vec{v}=\vec{i}-2\vec{k}$, and then explain the mathematic meaning of the above result.

(2) Using the gradient of $f(x,y,z)=2x^2+3y^2+z^2$ to find the divergence of grad *f*.
【91 中央電機(15%)】

【參考解答】

(1) $\left.\frac{df}{ds}\right|_{\vec{v}}=-\frac{4}{5}$ 含意爲：在點 P，順著 $\vec{v}$ 方向，單位長度內 f 減少 $\frac{4}{5}$ (2) $\nabla\cdot\nabla f=12$

習題 10

The temperature at a point (x,y) on a flat surface is given by $T(x,y)=100-2x^2-y^2$. Find the path a heat-seeking robot will take, starting at $(3,4)$, as it moves in the direction in which the temperature increases most rapidly.【1 中山機電(15%)】

【參考解答】$x=\frac{3}{16}y^2$ 爲所求

習題 11

如果一座山的高度 z 與水平座標 (x,y) 之關係爲 $z(x,y)=1500-6x^2-4y^2$ (單位：公尺)，且現今你所在山上位置的水平座標爲 (-10,-10)，

(1) 若你希望往最陡峭的方向前進，則此方向爲何？

(2) 若你由此位置向山頂方向前進，則須走多少公尺才能攻頂？

提示：$\int\sqrt{x^2+a^2}dx=\frac{1}{2}\left[x\sqrt{x^2+a^2}+a^2\log\left(x+\sqrt{x^2+a^2}\right)\right]$。【91 雲科營建(12%)】

【參考解答】

(1) $\nabla z=\frac{1}{\sqrt{13}}\left(3\vec{i}-2\vec{j}\right)$ 爲所求

(2) $s=10\left[10\sqrt{100+\frac{1}{200}}+\frac{1}{200}\ln\left(10+\sqrt{100+\frac{1}{200}}\right)\right]+\frac{1}{10}\ln 200$ 爲所求

習題 12

Consider the following mathematical expression

$$\sqrt{x^3y}+6x^2\sin\frac{y}{z}\pi=c \quad (c \text{ is a costant})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

It describes a surface in the 3-dimensional space. Why is that

(1) The gradient of the function is a vector?

(2) The gradient is normal to the surface?

$$\text{Consider}\quad f(x,y,z)=\sqrt{x^3y}+6x^2\sin\frac{y}{z}\pi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

(3) What is the directional derivative $\frac{df}{ds}$ of this function at $x=1$，$y=2$，$z=3$ and in the direction $\hat{i}+2\hat{j}+3\hat{k}$.

(4) What is the differential increment df of this function at $x=1$，$y=2$，$z=3$, for $\Delta x=10^{-3}$，$\Delta y=2\times10^{-3}$ and $\Delta z=3\times10^{-3}$?

(5) Can you calculate the same differential increment *df* in (4) for the surface described in EQ. 1?【90 中央光電(15%)】

【參考解答】

(1) ∇f 爲一向量

(2) ∇f 爲曲面法向量

(3) $\left.\frac{df}{ds}\right|_{\vec{e}}=\frac{1}{\sqrt{14}}\left(2\sqrt{2}+6\sqrt{3}\right)$

(4) $df=\left(2\sqrt{2}+6\sqrt{3}\right)\times10^{-3}$

(5) $\frac{df}{ds}ds=\frac{1}{\sqrt{14}}\left(2\sqrt{2}+6\sqrt{3}\right)\cdot\sqrt{14}\cdot10^{-3}$

習題 13

Find the directional derivative of $f(x,y)=x^4-3x^3y+x^2y^2$ at $(2,1)$ along the curve $x=t^2+1$, $y=t^3$ in the direction of increasing *t*.【89 成大土木(15%)】

【參考解答】$\left.\dfrac{df}{ds}\right)_{\vec{e}} = -\dfrac{48}{\sqrt{13}}$

習題 14

設 $\phi = \phi(u,v)$，$u = u(x,y,z)$，$v = v(x,y,z)$則 $\nabla\phi = \dfrac{\partial\phi}{\partial u}\nabla u + \dfrac{\partial\phi}{\partial v}\nabla v$。

【參考解答】$\nabla\phi - \dfrac{\partial\phi}{\partial u}\nabla u - \dfrac{\partial\phi}{\partial v}\nabla v = 0$同理，若$\phi = \phi(u)$，則$\nabla\phi = \phi'(u)\nabla u$

習題 15

The directional derivative of a function $\phi(x,y)$ at (x_0, y_0) in the direction of a unit vector $\vec{a}_u = \vec{a}_x u_x + \vec{a}_y u_y$ is defined by $\dfrac{d\phi}{d\vec{a}_u} = \lim\limits_{h\to 0}\dfrac{\phi(x_0 + hu_x, y_0 + hu_y) - \phi(x_0, y_0)}{h}$

Express this quantity in terms of $\vec{a}_u$ and $\nabla\phi(x_0, y_0)$. Show your derivative.【90 中山通訊(15%)】

【參考解答】$\dfrac{d\phi}{d\vec{a}_u} = \nabla\phi\cdot\vec{a}_u$

習題 16

若令$\nabla = \dfrac{\partial}{\partial x}\vec{i} + \dfrac{\partial}{\partial y}\vec{j} + \dfrac{\partial}{\partial z}\vec{k}$，$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$，$r = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$，則$\nabla r = \dfrac{\vec{r}}{r}$，

$\nabla\cdot\vec{r} = 3$，$\nabla\times\vec{r} = 0$，$\nabla\vec{r} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}$。

【參考解答】$\nabla r = \dfrac{\vec{r}}{r}$，$\nabla\cdot\vec{r} = 3$，$\nabla\times\vec{r} = 0$，$\nabla\vec{r} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}$，$\nabla\vec{r}$類似單位矩陣，任何向量與其內積，皆爲自己本身向量

習題 17

試證明$\vec{A}\cdot(\vec{B}\vec{C}) = (\vec{A}\cdot\vec{B})\vec{C}$，$(\vec{B}\vec{C})\cdot\vec{A} = \vec{B}(\vec{C}\cdot\vec{A})$。

【參考解答】$\vec{A}\cdot(\vec{B}\vec{C}) = (\vec{A}\cdot\vec{B})\vec{C}$，同理，$(\vec{B}\vec{C})\cdot\vec{A} = \vec{B}(\vec{C}\cdot\vec{A})$

習題 18

試證明$\overline{A}\cdot\left(\nabla\overline{B}\right)=\left(\overline{A}\cdot\nabla\right)\overline{B}$。

【參考解答】$\overline{A}\cdot\left(\nabla\overline{B}\right)=\left(\overline{A}\cdot\nabla\right)\overline{B}$，其中$\overline{A}\cdot\nabla=a_1\dfrac{\partial}{\partial x}+a_2\dfrac{\partial}{\partial y}+a_3\dfrac{\partial}{\partial z}$

習題 19

試說明$\nabla\overline{B}\cdot\overline{A}\neq\overline{A}\cdot\left(\nabla\overline{B}\right)$，$\nabla\cdot\overline{B}\neq\overline{B}\cdot\nabla$。

【參考解答】

$\nabla\overline{B}\cdot\overline{A}$表$\overline{B}$被微分，$\overline{B}$方向與$\overline{A}$方向內積，最後結果爲$\nabla$方向。$\overline{A}\cdot\left(\nabla\overline{B}\right)$表$\overline{B}$被微分，$\overline{A}$方向與$\nabla$方向內積，最後結果爲$\overline{B}$方向。$\nabla\cdot\overline{B}$表$\overline{B}$被微分，且$\overline{B}$方向和$\nabla$方向內積，$\overline{B}\cdot\nabla$表$\overline{B}$不被微分，但$\overline{B}$方向和$\nabla$方向內積。

■ 運算子

習題 1

Prove $\nabla\times\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\nabla\cdot\overline{B}\right)-\overline{B}\left(\nabla\cdot\overline{A}\right)$.【89 逢甲機械(18%)】

【參考解答】$\nabla\times\left(\overline{A}\times\overline{B}\right)=\left(\overline{B}\cdot\nabla\right)\overline{A}-\left(\overline{A}\cdot\nabla\right)\overline{B}+\overline{A}\left(\nabla\cdot\overline{B}\right)-\overline{B}\left(\nabla\cdot\overline{A}\right)$

習題 2

Show $\nabla\times\left(\nabla\times\vec{v}\right)=\nabla\left(\nabla\cdot\vec{v}\right)-\left(\nabla\cdot\nabla\right)\vec{v}$.【88 台大生物環境(5%)】

【參考解答】$\nabla\times\left(\nabla\times\vec{v}\right)=grad\left(div\vec{v}\right)-\nabla^2\vec{v}$

習題 3

$\vec{F}$ is a continuous 3-Dimensional vector field whose component have continuous first and second partial derivatives, and ϕ is a scalar function in xyz plane with continuous first and second partial derivatives.

(1) Prove $\nabla\cdot\left(\vec{\nabla}\times\vec{F}\right)=0$.

(2) Prove $\vec{\nabla}\times\left(\nabla\phi\right)=0$.【90 台科電子(10%)】

【參考解答】

(1) $\nabla\cdot\left(\vec{\nabla}\times\vec{F}\right)=0$ (2) $\vec{\nabla}\times\left(\nabla\phi\right)=0$

習題 4

Prove $\nabla\cdot\left(\vec{f}\times\vec{g}\right)=\left(\nabla\times\vec{f}\right)\cdot\vec{g}-\vec{f}\cdot\left(\nabla\times\vec{g}\right)$

【參考解答】$\nabla\cdot\left(\vec{f}\times\vec{g}\right)=\left(\nabla\times\vec{f}\right)\cdot\vec{g}-\vec{f}\cdot\left(\nabla\times\vec{g}\right)$

習題 5

Let vector $\vec{r}(x,y,z)=\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$ and $r=\left(x^2+y^2+z^2\right)^{1/2}$. Find

(1) $\nabla(1/r)$, $r\neq 0$.

(2) $\nabla^2(1/r)$, $r\neq 0$

(3) $\nabla\cdot\left(\vec{r}/r^3\right)$

(4) $\nabla\times\left(r^n\vec{r}\right)$【90 清大工程科學(20%)】

【參考解答】

(1) $\nabla\frac{1}{r}=-\frac{\vec{r}}{r^3}$ (2) $\nabla^2\left(\frac{1}{r}\right)=0$ (3) $\nabla\cdot\left(\frac{\vec{r}}{r^3}\right)=0$ (4) $\nabla\times\left(r^n\vec{r}\right)=0$

習題 6

Let $\vec{a}$ be a constant vector, $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, $\vec{\nabla}$ be gradient operator, $\times$ be cross product, and $\cdot$ be dot product. Evaluate the followings:

(1) $\left(\vec{a}\times\vec{\nabla}\right)\times\vec{r}$

(2) $\vec{\nabla}\times\left[\left(\vec{r}\cdot\vec{r}\right)\vec{a}\right]$

(3) $\vec{\nabla}\cdot\left[\left(\vec{r}\cdot\vec{r}\right)\vec{a}\right]$

(4) $\vec{\nabla}\cdot\left(\vec{a}\times\vec{r}\right)$

(5 $\vec{a}\times(\vec{\nabla}\times\vec{r})$)【90 台大化工(20%)】

【參考解答】

(1) $(\vec{a}\times\nabla)\times\vec{r}=-2\vec{a}$

(2) $\vec{\nabla}\times(r^2\vec{a})=2\vec{r}\times\vec{a}$，其中 $r=|\vec{r}|$

(3) $\nabla\cdot(r^2\vec{a})=2\vec{r}\cdot\vec{a}$

(4) $\nabla\cdot(\vec{a}\times\vec{r})=0$

(5) $\vec{a}\times(\nabla\times\vec{r})=0$，$\because \nabla\times\vec{r}=0$

習題 7

Let $\vec{A}$ and f be vector and scalar fields, respectively.

(1) Prove $\nabla\cdot\nabla\times\vec{A}=0$.

(2) Express $\nabla\cdot(f\vec{A})$ in terms of f, $\vec{A}$, ∇f and $\nabla\cdot\vec{A}$.

(3) Suppose $\vec{A}=\vec{a}_x 2xy+\vec{a}_y x^2+\vec{a}_z z$. Find an f such that $\vec{A}=\nabla f$.【91 中山通訊 (15%)】

【參考解答】

(1) $\nabla\cdot(\nabla\times\vec{A})=0$，for $\vec{A}=\vec{a}_1 i+\vec{a}_2 j+\vec{a}_3 k$

(2) $\nabla\cdot(f\vec{A})=\nabla f\cdot\vec{A}+f(\nabla\cdot\vec{A})$

(3) $\nabla f=\vec{A}$，得 $f=x^2y+\frac{1}{2}z^2+c$

習題 8

Find the condition that minimizes the surface area A of a thin can (with base radius r and height h) of fixed volume V.【91 交大光電(10%)】

【參考解答】最小面積 $A=6\pi\left(\frac{V}{2\pi}\right)^{\frac{2}{3}}$

習題 9

Find the closest point on the plane $Ax+By+Cz+D=0$ to the point (x_0,y_0,z_0) in space. Formulate this problem as a constrained minimization problem.【90 中山海下(15%)】

【參考解答】在平面上，最接近(x_0,y_0,z_0)之點為(x,y,z)，

$x=x_0-\dfrac{A(Ax_0+By_0+Cz_0+D)}{A^2+B^2+C^2}$，$y=y_0-\dfrac{B(Ax_0+By_0+Cz_0+D)}{A^2+B^2+C^2}$，

$z=z_0-\dfrac{B(Ax_0+By_0+Cz_0+D)}{A^2+B^2+C^2}$

習題 10

Find the minimum of the function $f(x,y,z)=2xy+6yz+8xz$, subject to the constraint $xyz=12000$.【89 台大環工(15%)】

【參考解答】$\dfrac{6}{x}=\dfrac{8}{y}=\dfrac{2}{z}$，$\dfrac{x}{3}=\dfrac{y}{4}=\dfrac{z}{1}=t$，$x=3t$，$y=4t$，$z=t$，代入$xyz=12000$，$12t^3=12000$，$t=10$，$x=30$，$y=40$，$z=10$為極值發生點，$f=7200$為極小值

習題 11

(1)Find ∇f, $f=e^{t\vec{k}\cdot\vec{r}}$, $\vec{k}$ is a constant vector.

(2)If $\vec{k}=(1,2,4)$, find $\nabla\times(f\vec{v})$, where $\vec{v}$ is an unit constant vector and $\vec{v}\times\vec{k}=0$.

【87 中山光電(10%)】

【參考解答】(1) $\nabla f=te^{t\vec{k}\cdot\vec{r}}\vec{k}$ (2) $\nabla\times(f\vec{v})=0$

習題 12

$v=-(x\vec{i}+y\vec{j}+z\vec{k})(x^2+y^2+z^2)^{-\frac{3}{2}}$, find f for $\nabla f=\vec{v}$.【86 台大農機(15%)】

【參考解答】$f'(r)=-\dfrac{1}{r^2}$，$f(r)=\dfrac{1}{r}+c$，$f=(x^2+y^2+z^2)^{-\frac{1}{2}}+c$

習題 13

假設各函數有足夠的可微分(sufficient differentiability)，f 與 $\vec{v}$ 分別爲點位置座標的純量與向量函數，試根據 grad，div 與 curl 的定義，證明下列二式：

(1) $div\left(f\ \vec{v}\right)=f\ div\ \vec{v}+\vec{v}\cdot grad\ f$ 。

(2) $curl\left(f\ \vec{v}\right)=f\ \ curl\vec{v}+grad\ f\times\vec{v}$ 。【91 台大生物環境(10%)】

【參考解答】(1) $\nabla\cdot\left(f\vec{v}\right)=\vec{v}\cdot\nabla f+f\nabla\cdot\vec{v}$ (2) $curl\left(f\vec{v}\right)=f\ curl\ v+\nabla f\times\vec{v}$

習題 14

一內半徑爲 a 之圓球內裝一正圓柱，求此正圓柱最大可能之體積?【89 成大資源(15%)】

【參考解答】圓柱體最大體積$V=4\pi\left(\frac{a}{\sqrt{3}}\right)3$

習題 15

If $\vec{v}=\vec{v}\left(\vec{a}\cdot\vec{r}\right)$, where $\vec{a}$ is a constant vector and $\vec{r}$ is the radius vector evaluate

$\nabla\cdot\vec{v}$ and $\nabla\times\vec{v}$. 【91 台師大物理(20%)】

【參考解答】$\nabla\cdot\vec{v}=\vec{v}'\cdot\vec{a}$ ，$\nabla\times\vec{v}=\vec{a}\times\vec{v}'$

習題 16

求曲線 $x^2+y^2=1$ ，$x^2-xy+y^2-z^2=1$ 上，距離原點最近之點。【90 交大電物(20%)】

【參考解答】$\left(x,y,z\right)=\left(1,0,0\right)$ ，$\left(0,1,0\right)$ ，$\left(-1,0,0\right)$ ，$\left(0,-1,0\right)$

習題 17

Find the point (x, y, z) at which the function $f(x, y, z) = z$, subject to the constrains $x + y + z = 1$ and $\frac{z^2}{xy^3} = 3$, is maximized. Evaluate the maximum of $f(x, y, z)$.【91 中原物理(20%)】

【參考解答】 f 之極大值 $f = z = \frac{17}{9} + \frac{4\sqrt{13}}{9}$，發生於 $y = -\frac{2}{3} - \frac{\sqrt{13}}{3}$，$z = \frac{17}{9} + \frac{4\sqrt{13}}{9}$，$x = -\frac{2}{9} - \frac{1}{9}\sqrt{13}$

■ 曲線座標

習題 1

圓柱座標 $x = r\cos\theta$，$y = r\sin\theta$，$z = z$，求

(1) h_r, h_θ, h_z (2) $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$

【參考解答】

(1) $h_r = 1$，$h_\theta = r$，$h_z = 1$

(2) $\vec{e}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$，$\vec{e}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$，$\vec{e}_z = \vec{k}$

習題 2

球座標 $x = \rho\sin\theta\cos\phi$，$y = \rho\sin\theta\cos\phi$，$z = \rho\cos\theta$，求

(1) h_ρ，h_θ，h_ϕ (2) $\vec{e}_\rho$，$\vec{e}_\theta$，$\vec{e}_\phi$

【參考解答】

(1) $h_\rho = 1$，$h_\theta = -\sin\phi\vec{i} + \cos\phi\vec{j}$，$h_\phi = \rho\sin\theta$

(2) $\vec{e}_\rho = \frac{\vec{r}}{\rho}$，$\vec{e}_\theta = \cos\theta\cos\phi\vec{i} + \cos\theta\sin\phi\vec{j} - \sin\theta\vec{k}$，$\vec{e}_\phi = -\sin\phi\vec{i} + \cos\phi\vec{j}$，

習題 3

在球座標中，若 $\overline{A} = A_r\vec{e}_r + A_g\vec{e}_\theta + \overline{A}_\phi\vec{e}_\phi$，$B$ 爲純量函數。求：

(1) $\nabla \cdot \overline{A}$ (2) $\nabla \times \overline{A}$ (3) ∇B (4) $\nabla^2 B$

【參考解答】

(1) $\nabla\cdot\overline{A}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot A_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

(2) $\nabla\times\overline{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$

(3) $\nabla B=\frac{\partial B}{h_r\partial r}\vec{e}_r+\frac{\partial B}{h_\theta\partial\theta}\vec{e}_\theta+\frac{\partial B}{h_\phi\partial\phi}\vec{e}_\phi$

(4) $\nabla^2B=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial B}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial B}{\partial\theta}\sin\theta\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2B}{\partial\phi^2}$

習題 4

在極座標中，若$\overline{A}=A_r\vec{e}_r+A_\theta\vec{e}_\theta+A_z\vec{e}_z$ ，B爲純量函數。求

(1)$\nabla\cdot\overline{A}$ (2)$\nabla\times\overline{A}$ (3)∇B (4)∇^2B

【參考解答】

(1) $\nabla\cdot\overline{A}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot A_r\right)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_z}{\partial z}$

(2) $\nabla\times\overline{A}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & \overline{e_z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix}$

(3) $\nabla B=\frac{\partial B}{\partial r}\bar{e}_r+\frac{1}{r}\frac{\partial B}{\partial\theta}\bar{e}_\theta+\frac{\partial B}{\partial z}\bar{e}_z$

(4) $\nabla^2B=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial B}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2B}{\partial\theta^2}+\frac{\partial^2B}{\partial z^2}$

習題 5

Compute $\iint_s\overline{F}\cdot\vec{n}d\sigma$, where $\overline{F}(x,y,z)=x^2\hat{i}+2y\hat{j}+4z^2\hat{k}$ and s is the surface of the cylinder $x^2+y^2\le 4$，$0\le z\le 2$. 【89 台科電子(12%)】

【參考解答】$\iint_s\overline{F}\cdot\vec{n}dA=80\pi$

習題 6

Determine the surface integral of function $f = x^2 z$ over the entire surface of the circular cylinder of height h which stands on the circle $x^2 + y^2 = 9$ as shown in the following figure. 【89 交大機械(20%)】

【參考解答】$\iint_s f dA = \frac{27}{2}\pi h^2$

習題 7

Find $\iiint_R 5x^2 dv$, R is the cylinder of $0 \le z \le h$, $x^2 + y^2 \le 4$. 【89 成大工程科學(14%)】

【參考解答】$\iiint_R 5x^2 dv = 20\pi h$

習題 8

Evaluate $\iint_s \vec{F} \cdot \vec{n} d\sigma$, where $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$ and s is lateral surface of the cylinder $x^2 + y^2 = 9$, $0 \le z \le 1$. 【91 中原化工(10%)】

【參考解答】$\iint_S \vec{F} \cdot \vec{n} dA = 27\pi$

習題 9

Let $\vec{f} = rz\vec{e}_r + 3\vec{e}_\theta + rz^2\vec{e}_z$, find the surface integral $\int_S \vec{n} \cdot \vec{f} dA$ including bottom, top and side for a cylinder $0 \le r \le 3$, $0 \le z \le 6$. 【89 交大機械(17%)】

【參考解答】$\iint \vec{f} \cdot \vec{n} dA = 972\pi$