提要 299：三個函數的 Jacobian 問題

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已知：

$$f(x,y,z,u,v,w)=C_1$$
$$g(x,y,z,u,v,w)=C_2$$
$$h(x,y,z,u,v,w)=C_3$$

試推求 $\dfrac{\partial u}{\partial x}$、$\dfrac{\partial v}{\partial y}$、$\dfrac{\partial w}{\partial z}$。

解答：

首先對 $f(x,y,z,u,v,w)=C_1$、$g(x,y,z,u,v,w)=C_2$、$h(x,y,z,u,v,w)=C_3$ 作全微分(Total Differential)之運算，亦即：

$$\begin{cases} df(x,y,z,u,v,w)=dC_1 & (1a) \\ dg(x,y,z,u,v,w)=dC_2 & (1b) \\ dh(x,y,z,u,v,w)=dC_3 & (1c) \end{cases}$$

上式可改寫為：

$$\begin{cases} \dfrac{\partial f}{\partial x}dx+\dfrac{\partial f}{\partial y}dy+\dfrac{\partial f}{\partial z}dz+\dfrac{\partial f}{\partial u}du+\dfrac{\partial f}{\partial v}dv+\dfrac{\partial f}{\partial w}dw=0 & (2a) \\ \dfrac{\partial g}{\partial x}dx+\dfrac{\partial g}{\partial y}dy+\dfrac{\partial g}{\partial z}dz+\dfrac{\partial g}{\partial u}du+\dfrac{\partial g}{\partial v}dv+\dfrac{\partial g}{\partial w}dw=0 & (2b) \\ \dfrac{\partial h}{\partial x}dx+\dfrac{\partial h}{\partial y}dy+\dfrac{\partial h}{\partial z}dz+\dfrac{\partial h}{\partial u}du+\dfrac{\partial h}{\partial v}dv+\dfrac{\partial h}{\partial w}dw=0 & (2c) \end{cases}$$

由式(2a)-(2c)分別對 x 、 y 、 z 微分，可分別推求出：

$$\begin{cases} \dfrac{\partial f}{\partial x}\dfrac{dx}{dx}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial x}+\dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial x}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}=0 & (3a) \\ \dfrac{\partial g}{\partial x}\dfrac{dx}{dx}+\dfrac{\partial g}{\partial y}\dfrac{\partial y}{\partial x}+\dfrac{\partial g}{\partial z}\dfrac{\partial z}{\partial x}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial x}=0 & (3b) \\ \dfrac{\partial h}{\partial x}\dfrac{dx}{dx}+\dfrac{\partial h}{\partial y}\dfrac{\partial y}{\partial x}+\dfrac{\partial h}{\partial z}\dfrac{\partial z}{\partial x}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial x}=0 & (3c) \end{cases}$$

$$
\begin{cases}
\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial y}+\dfrac{\partial f}{\partial y}\dfrac{dy}{dy}+\dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(4a)}\\
\dfrac{\partial g}{\partial x}\dfrac{\partial x}{\partial y}+\dfrac{\partial g}{\partial y}\dfrac{dy}{dy}+\dfrac{\partial g}{\partial z}\dfrac{\partial z}{\partial y}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(4b)}\\
\dfrac{\partial h}{\partial x}\dfrac{\partial x}{\partial y}+\dfrac{\partial h}{\partial y}\dfrac{dy}{dy}+\dfrac{\partial h}{\partial z}\dfrac{\partial z}{\partial y}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(4c)}
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial z}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial z}+\dfrac{\partial f}{\partial z}\dfrac{dz}{dz}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial z}=0 & \text{(5a)}\\
\dfrac{\partial g}{\partial x}\dfrac{\partial x}{\partial z}+\dfrac{\partial g}{\partial y}\dfrac{\partial y}{\partial z}+\dfrac{\partial g}{\partial z}\dfrac{dz}{dz}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial z}=0 & \text{(5b)}\\
\dfrac{\partial h}{\partial x}\dfrac{\partial x}{\partial z}+\dfrac{\partial h}{\partial y}\dfrac{\partial y}{\partial z}+\dfrac{\partial h}{\partial z}\dfrac{dz}{dz}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(5c)}
\end{cases}
$$

由題意知，(u,v,w)是一組獨立變數、(x,y,z)是另一組獨立變數，故$\dfrac{\partial y}{\partial x}=0$、$\dfrac{\partial x}{\partial y}=0$、$\dfrac{\partial y}{\partial z}=0$、$\dfrac{\partial z}{\partial y}=0$、$\dfrac{\partial z}{\partial x}=0$、$\dfrac{\partial x}{\partial z}=0$。基於此，式(3a)-(3c)、(4a)-(4c)、(5a)-(5c)亦可表為：

$$
\begin{cases}
\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}=0 & \text{(6a)}\\
\dfrac{\partial g}{\partial x}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial x}=0 & \text{(6b)}\\
\dfrac{\partial h}{\partial x}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial x}=0 & \text{(6c)}
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(7a)}\\
\dfrac{\partial g}{\partial y}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(7b)}\\
\dfrac{\partial h}{\partial y}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial y}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(7c)}
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial f}{\partial z}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial z}=0 & \text{(8a)}\\
\dfrac{\partial g}{\partial z}+\dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial g}{\partial w}\dfrac{\partial w}{\partial z}=0 & \text{(8b)}\\
\dfrac{\partial h}{\partial z}+\dfrac{\partial h}{\partial u}\dfrac{\partial u}{\partial z}+\dfrac{\partial h}{\partial v}\dfrac{\partial v}{\partial z}+\dfrac{\partial h}{\partial w}\dfrac{\partial w}{\partial y}=0 & \text{(8c)}
\end{cases}
$$

為簡化符號，各項微分式亦可分別表為$\dfrac{\partial f}{\partial x}=f_x$、$\dfrac{\partial f}{\partial y}=f_y$、$\dfrac{\partial f}{\partial z}=f_z$、$\dfrac{\partial f}{\partial u}=f_u$、$\dfrac{\partial f}{\partial v}=f_v$、

$\frac{\partial f}{\partial w}=f_w$、$\frac{\partial g}{\partial x}=g_x$、$\frac{\partial g}{\partial y}=g_y$、$\frac{\partial g}{\partial z}=g_z$、$\frac{\partial g}{\partial u}=g_u$、$\frac{\partial g}{\partial v}=g_v$、$\frac{\partial g}{\partial w}=g_w$，則式(6a)-(6c)、(7a)-(7c)、(8a)-(8c)又可改寫為：

$$\begin{cases} f_u\frac{\partial u}{\partial x}+f_v\frac{\partial v}{\partial x}+f_w\frac{\partial w}{\partial x}=-f_x & (9a)\\ g_u\frac{\partial u}{\partial x}+g_v\frac{\partial v}{\partial x}+g_w\frac{\partial w}{\partial x}=-g_x & (9b)\\ h_u\frac{\partial u}{\partial x}+h_v\frac{\partial v}{\partial x}+h_w\frac{\partial w}{\partial x}=-h_x & (9c)\end{cases}$$

$$\begin{cases} f_u\frac{\partial u}{\partial y}+f_v\frac{\partial v}{\partial y}+f_w\frac{\partial w}{\partial y}=-f_y & (10a)\\ g_u\frac{\partial u}{\partial y}+g_v\frac{\partial v}{\partial y}+g_w\frac{\partial w}{\partial y}=-g_y & (10b)\\ h_u\frac{\partial u}{\partial y}+h_v\frac{\partial v}{\partial y}+h_w\frac{\partial w}{\partial y}=-h_y & (10c)\end{cases}$$

$$\begin{cases} f_u\frac{\partial u}{\partial z}+f_v\frac{\partial v}{\partial z}+f_w\frac{\partial w}{\partial z}=-f_z & (11a)\\ g_u\frac{\partial u}{\partial z}+g_v\frac{\partial v}{\partial z}+g_w\frac{\partial w}{\partial z}=-g_z & (11b)\\ h_u\frac{\partial u}{\partial z}+h_v\frac{\partial v}{\partial z}+h_w\frac{\partial w}{\partial y}=-h_z & (11c)\end{cases}$$

式(9a)-(9c)、式(10a)-(10c)與式(11a)-(11c)可分別表為：

$$\begin{bmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{bmatrix}=\begin{bmatrix} -f_x \\ -g_x \\ -h_x \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial y} \end{bmatrix}=\begin{bmatrix} -f_y \\ -g_y \\ -h_y \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} -f_z \\ -g_z \\ -h_z \end{bmatrix} \tag{14}$$

應用 Cramer 定則解析式(12)，即可推求出 $\frac{\partial u}{\partial x}$ 之解：

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -f_x & f_v & f_w \\ -g_x & g_v & g_w \\ -h_x & h_v & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{\begin{vmatrix} f_x & f_v & f_w \\ g_x & g_v & g_w \\ h_x & h_v & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{J\left(\frac{f,g,h}{x,v,w}\right)}{J\left(\frac{f,g,h}{u,v,w}\right)} \tag{15}$$

同理，應用 Cramer 定則解析式(13)，即可推求出 $\frac{\partial v}{\partial y}$ 之解：

$$\frac{\partial v}{\partial y} = \frac{\begin{vmatrix} f_u & -f_y & f_w \\ g_u & -g_y & g_w \\ h_u & -h_y & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{\begin{vmatrix} f_u & f_y & f_w \\ g_u & g_y & g_w \\ h_u & h_y & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{J\left(\frac{f,g,h}{u,y,w}\right)}{J\left(\frac{f,g,h}{u,v,w}\right)} \tag{16}$$

同理，應用 Cramer 定則解析式(14)，即可推求出 $\frac{\partial w}{\partial z}$ 之解：

$$\frac{\partial w}{\partial z} = \frac{\begin{vmatrix} f_u & f_v & -f_z \\ g_u & g_v & -g_z \\ h_u & h_v & -h_z \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{\begin{vmatrix} f_u & f_v & f_z \\ g_u & g_v & g_z \\ h_u & h_v & h_z \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = -\frac{J\left(\frac{f,g,h}{u,v,z}\right)}{J\left(\frac{f,g,h}{u,v,w}\right)} \tag{17}$$

附註：三個函數 F 、G 、H 之 Jacobian 是定義為 $J\left(\frac{F,G,H}{\alpha,\beta,\gamma}\right)=\begin{vmatrix} F_\alpha & F_\beta & F_\gamma \\ G_\alpha & G_\beta & G_\gamma \\ H_\alpha & H_\beta & H_\gamma \end{vmatrix}$ ，其中

$F_\alpha=\frac{\partial F}{\partial\alpha}$ 、$F_\beta=\frac{\partial F}{\partial\beta}$ 、$F_\gamma=\frac{\partial F}{\partial\gamma}$ 、$G_\alpha=\frac{\partial G}{\partial\alpha}$ 、$G_\beta=\frac{\partial G}{\partial\beta}$ 、$G_\gamma=\frac{\partial G}{\partial\gamma}$ 、$H_\alpha=\frac{\partial H}{\partial\alpha}$ 、$H_\beta=\frac{\partial H}{\partial\beta}$ 、$H_\gamma=\frac{\partial H}{\partial\gamma}$ 。