提要 295：圓柱座標系統的 Laplacian

Laplacian ∇^2 也是一個專有名詞，在 (x,y,z) 卡氏座標系統(Cartesian Coordinates)中，其係定義為 $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ 。本單元旨在解釋 (r,θ,z) 圓柱座標系統(Cylindrical Coordinates)下之 Laplacian 表示法。

圓柱座標系統的 Laplacian

如圖 1 所示圓柱座標系統 (r,θ,z) 的 Laplacian ∇^2 可表為：

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

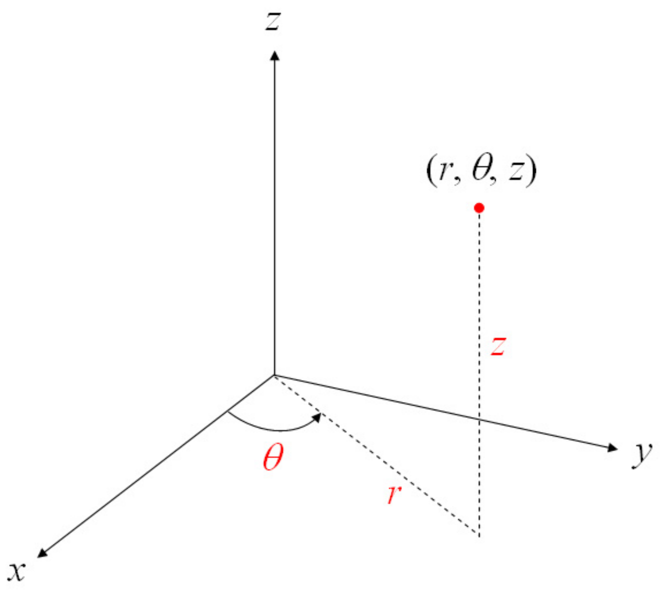

圖 1　圓柱座標系統 (r,θ,z) 的表示法

證明：

觀察圖 1 知，卡氏座標系統(x,y,z)與圓柱座標系統(r,θ,z)有以下之關係：

$$x = r\cos\theta \text{、} y = r\sin\theta \text{、} z = z \tag{1}$$

故

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \tag{2a}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \tag{2b}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z} = \frac{\partial}{\partial z} \tag{2c}$$

式(2a)-(2c)可整理為：

$$\begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \tag{3}$$

上式之反轉換可表為：

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\dfrac{1}{r}\sin\theta & 0 \\ \sin\theta & \dfrac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \tag{4}$$

亦即：

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \tag{5a}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta} \tag{5b}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{5c}$$

所以：

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \\
&= \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)+\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right)+\left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right) \\
&= \left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&+\left(\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)+\left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}-\frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} \\
&+\frac{1}{r}\sin^2\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}+\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2} \\
&+\sin^2\theta\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} \\
&+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}+\frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2} \\
&+\frac{\partial^2}{\partial z^2} \\
&= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}
\end{aligned}
$$

故得證。