提要 287：一維波傳問題之 D'Alembert 解答

一維波傳問題之 D'Alembert 解答

如圖 1 所示弦索振動問題之數學模式為：

■ 控制方程式：$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ 【牛頓第二運動定律的化身】，其中 $u(x,t)$ 表質點之位移量；係數 $c=T/\rho$ 是波傳速度，T、ρ 分別為弦索拉力與線密度。

■ 邊界條件：$u(0,t)=0$ 【弦索左邊為固定端】

$u(L,t)=0$ 【弦索右邊為固定端】

■ 初始條件：$u(x,0)=f(x)$ 【弦索之初始形狀為 $f(x)$】

$\dfrac{\partial u(x,0)}{\partial t}=g(x)$ 【弦索之初始速度為 $g(x)$】

則其 **D'Alembert 解答**為

$$\begin{cases} u(x,t)=\phi(x+ct)+\psi(x-ct) \\ u(x,t)=\dfrac{1}{2}\left[f(x+ct)+f(x-ct)+\dfrac{1}{c}\displaystyle\int_{x-ct}^{x+ct}g(s)ds\right] \end{cases}$$

，其中之第一式尚未考慮問題之初始條件，第二式則有考慮問題之初始條件。

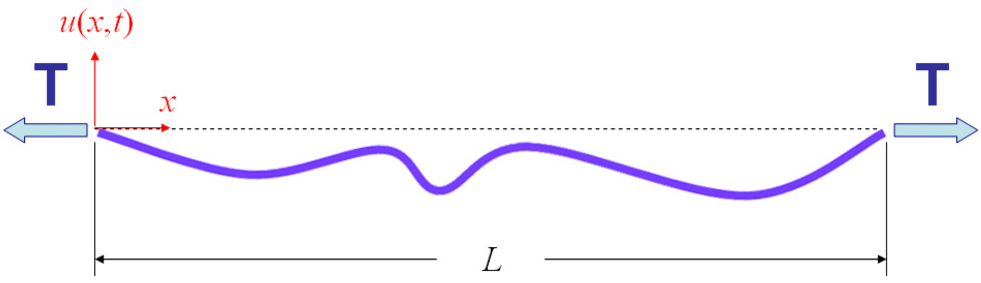

圖 1　弦索振動問題示意圖

證明：

關於本單元之說明似已超過一般研究所的考試範圍，若情況不允許，吾人認為可以略過不讀。已知一維波傳方程式為：

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

若作變數變換，考慮：

$$v=x+ct \text{ 、 } z=x-ct \qquad (2)$$

則

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial u}{\partial v}(1)+\frac{\partial u}{\partial z}(1)=\frac{\partial u}{\partial v}+\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}=\frac{\partial u}{\partial v}(c)+\frac{\partial u}{\partial z}(-c)=c\left(\frac{\partial u}{\partial v}-\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial z}\right)}{\partial v}(1)+\frac{\partial\left(\frac{\partial u}{\partial v}+\frac{\partial u}{\partial z}\right)}{\partial z}(1)=\frac{\partial^2 u}{\partial v^2}+2\frac{\partial^2 u}{\partial v\partial z}+\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial\left(\frac{\partial u}{\partial t}\right)}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial\left(\frac{\partial u}{\partial t}\right)}{\partial z}\frac{\partial z}{\partial t}=\frac{\partial\left(c\frac{\partial u}{\partial v}-c\frac{\partial u}{\partial z}\right)}{\partial v}(c)+\frac{\partial\left(c\frac{\partial u}{\partial v}-c\frac{\partial u}{\partial z}\right)}{\partial z}(-c)=c^2\left(\frac{\partial^2 u}{\partial v^2}-2\frac{\partial^2 u}{\partial v\partial z}+\frac{\partial^2 u}{\partial z^2}\right)$$

將 $\frac{\partial^2 u}{\partial x^2}$ 與 $\frac{\partial^2 u}{\partial t^2}$ 之結果代回式(1)，則式(1)可改寫為：

$$\frac{\partial^2 u}{\partial v^2}+2\frac{\partial^2 u}{\partial v\partial z}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\cdot c^2\left(\frac{\partial^2 u}{\partial v^2}-2\frac{\partial^2 u}{\partial v\partial z}+\frac{\partial^2 u}{\partial z^2}\right)$$

故上式可化簡為：

$$\frac{\partial^2 u}{\partial v\partial z}=0 \qquad (3)$$

式(3)對變數 z 作積分，可得：

$$\frac{\partial u}{\partial v}=h(v) \qquad (4)$$

式(4)再對變數 v 作積分，可得：

$$u=\int h(v)dv+\psi(z)=\phi(v)+\psi(z) \tag{5}$$

再將式(2)代入式(5)，則式(5)可表為：

$$\boxed{u(x,t)=\phi(x+ct)+\psi(x-ct)} \tag{6}$$

上式即為式(1)之 D'Alembert 解答，式(6)尚未考慮問題之初始條件。

已知問題之初始條件為：

$$\begin{cases} u(x,0)=f(x) \\ \dfrac{\partial u(x,0)}{\partial t}=g(x) \end{cases} \tag{7}$$

將式(6)代入式(7)可得：

$$\begin{cases} \phi(x)+\psi(x)=f(x) \\ c\phi'(x)-c\psi'(x)=g(x) \end{cases} \tag{8}$$

解析式(8)之第二式可得：

$$\phi(x)-\psi(x)=\int\frac{1}{c}g(x)dx+常數=\int_{x_0}^{x}\frac{1}{c}g(s)ds+\phi(x_0)-\psi(x_0) \tag{9}$$

上式與(8)之第一式聯立可解出：

$$\begin{aligned} \phi(x)&=\frac{1}{2}\left[f(x)+\frac{1}{c}\int_{x_0}^{x}g(s)ds+k(x_0)\right] \\ \psi(x)&=\frac{1}{2}\left[f(x)-\frac{1}{c}\int_{x_0}^{x}g(s)ds-k(x_0)\right] \end{aligned} \tag{10}$$

其中 $k(x_0)=\phi(x_0)-\psi(x_0)$。式(10)中之第一式的變數 x 換成 $x+ct$ 、第二式的變數 x 換成 $x-ct$ ，然後再作相加之運算，則：

$$
\begin{aligned}
\phi(x+ct)+\psi(x-ct) &= \frac{1}{2}\left[f(x+ct)+f(x-ct)+\frac{1}{c}\int_{x_0}^{x+ct} g(s)ds + k(x_0) - \frac{1}{c}\int_{x_0}^{x-ct} g(s)ds - k(x_0)\right] \\
&= \frac{1}{2}\left[f(x+ct)+f(x-ct)+\frac{1}{c}\int_{x_0}^{x+ct} g(s)ds + \frac{1}{c}\int_{x-ct}^{x_0} g(s)ds\right] \\
&= \frac{1}{2}\left[f(x+ct)+f(x-ct)+\frac{1}{c}\int_{x-ct}^{x+ct} g(s)ds\right]
\end{aligned}
$$

上式即為有考慮初始條件下之 D’Alembert 解答，故得證。