提要 235：曲線座標系統之 Laplacian

Laplacian ∇^2 是一個專有名詞，在 (x,y,z) 卡氏座標系統(Cartesian Coordinates)中，其係定義為 $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$。本單元旨在解釋曲線座標系統下之 Laplacian 表示法。

曲線座標系統之 Laplacian

若 (x_1,x_2,x_3) 表卡氏座標系統、(q_1,q_2,q_3) 表曲線直角座標系統，則曲線座標系統之 Laplacian $\nabla^2 f$ 為：

$$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{1}{h_3}\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right] \tag{1}$$

其中 $h_1^2=\left(\dfrac{\partial x_1}{\partial q_1}\right)^2+\left(\dfrac{\partial x_2}{\partial q_1}\right)^2+\left(\dfrac{\partial x_3}{\partial q_1}\right)^2$，

$h_2^2=\left(\dfrac{\partial x_1}{\partial q_2}\right)^2+\left(\dfrac{\partial x_2}{\partial q_2}\right)^2+\left(\dfrac{\partial x_3}{\partial q_2}\right)^2$，

$h_3^2=\left(\dfrac{\partial x_1}{\partial q_3}\right)^2+\left(\dfrac{\partial x_2}{\partial q_3}\right)^2+\left(\dfrac{\partial x_3}{\partial q_3}\right)^2$。

【證明】

茲令向量函數 $\mathbf{F}=F_1\mathbf{u}+F_2\mathbf{v}+F_3\mathbf{w}$ 為純量函數 f 的梯度：

$$\mathbf{F}=\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{u}+\frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{v}+\frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{w} \tag{2}$$

則 $F_1=\dfrac{1}{h_1}\dfrac{\partial f}{\partial q_1}$、$F_2=\dfrac{1}{h_2}\dfrac{\partial f}{\partial q_2}$、$F_3=\dfrac{1}{h_3}\dfrac{\partial f}{\partial q_3}$。又已知向量函數 $\mathbf{F}$ 的散度為：

$$\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial q_1}+\frac{1}{h_2}\frac{\partial(h_1h_3F_2)}{\partial q_2}+\frac{1}{h_3}\frac{\partial(h_1h_2F_3)}{\partial q_3}\right] \tag{3}$$

將式(2)代入式(3)：

$$\nabla \cdot \mathbf{F} = \nabla \cdot \nabla f$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 F_1)}{\partial q_1} + \frac{1}{h_2}\frac{\partial(h_1 h_3 F_2)}{\partial q_2} + \frac{1}{h_3}\frac{\partial(h_1 h_2 F_3)}{\partial q_3}\right]$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{1}{h_2}\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{1}{h_3}\frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

故得證。

【附註】

■ 圓柱座標系統(r,θ,z)的 Laplacian ∇^2 可表為(請參閱提要 295)：

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

■ 球體座標系統(r,θ,ϕ)的 Laplacian ∇^2 可表為(請參閱提要 296)：

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$