提要 234：曲線座標系統之散度

曲線座標系統之散度

若(x_1, x_2, x_3)表卡氏座標系統、(q_1, q_2, q_3)表曲線直角座標系統，則曲線座標系統之散度$\nabla\cdot\mathbf{F}$為：

$$\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial q_1}+\frac{1}{h_2}\frac{\partial(h_1h_3F_2)}{\partial q_2}+\frac{1}{h_3}\frac{\partial(h_1h_2F_3)}{\partial q_3}\right] \tag{1}$$

其中 $h_1^2=\left(\frac{\partial x_1}{\partial q_1}\right)^2+\left(\frac{\partial x_2}{\partial q_1}\right)^2+\left(\frac{\partial x_3}{\partial q_1}\right)^2$；

$h_2^2=\left(\frac{\partial x_1}{\partial q_2}\right)^2+\left(\frac{\partial x_2}{\partial q_2}\right)^2+\left(\frac{\partial x_3}{\partial q_2}\right)^2$；

$h_3^2=\left(\frac{\partial x_1}{\partial q_3}\right)^2+\left(\frac{\partial x_2}{\partial q_3}\right)^2+\left(\frac{\partial x_3}{\partial q_3}\right)^2$；

$\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ 。

【證明】

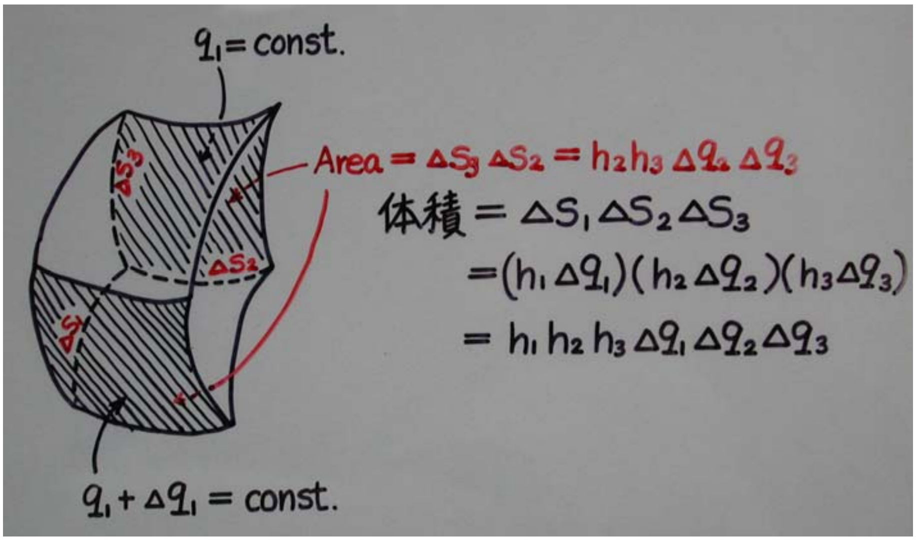

圖 1　通過曲邊立方體六個面的流量之思考方式

散度是流量的概念，所以通過如圖 1 所示曲邊立方體(其邊長分別為Δs_1、Δs_2、Δs_3）

之六個面的流量之和為：

$$
\begin{aligned}
\nabla\cdot\mathbf{F} &= \text{Flux} \\
&= \text{Flux in }\mathbf{u}+\text{Flux in }\mathbf{v}+\text{Flux in }\mathbf{w} \\
&= \{F_1(q_1+\Delta q_1,q_2,q_3)[h_2h_3\Delta q_2\Delta q_3]-F_1(q_1,q_2,q_3)[h_2h_3\Delta q_2\Delta q_3] \\
&+F_2(q_1,q_2+\Delta q_2,q_3)[h_1h_3\Delta q_1\Delta q_3]-F_2(q_1,q_2,q_3)[h_1h_3\Delta q_1\Delta q_3] \\
&+F_3(q_1,q_2,q_3+\Delta q_3)[h_1h_2\Delta q_1\Delta q_2]-F_3(q_1,q_2,q_3)[h_1h_2\Delta q_1\Delta q_2]\} \\
&\div h_1h_2h_3\Delta q_1\Delta q_2\Delta q_3 \\
&= \frac{\Delta q_2\Delta q_3\{[F_1(q_1+\Delta q_1,q_2,q_3)h_2h_3]-[F_1(q_1,q_2,q_3)h_2h_3]\}}{h_1h_2h_3\Delta q_1\Delta q_2\Delta q_3} \\
&+\frac{\Delta q_1\Delta q_3\{[F_2(q_1,q_2+\Delta q_2,q_3)h_1h_3]-[F_2(q_1,q_2,q_3)h_1h_3]\}}{h_1h_2h_3\Delta q_1\Delta q_2\Delta q_3} \\
&+\frac{\Delta q_1\Delta q_2\{[F_3(q_1,q_2,q_3+\Delta q_3)h_1h_2]-[F_3(q_1,q_2,q_3)h_1h_2]\}}{h_1h_2h_3\Delta q_1\Delta q_2\Delta q_3} \\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial q_1}+\frac{1}{h_2}\frac{\partial(h_1h_3F_2)}{\partial q_2}+\frac{1}{h_3}\frac{\partial(h_1h_2F_3)}{\partial q_3}\right]
\end{aligned} \qquad (2)
$$

其中 $h_1^2=\left(\frac{\partial x_1}{\partial q_1}\right)^2+\left(\frac{\partial x_2}{\partial q_1}\right)^2+\left(\frac{\partial x_3}{\partial q_1}\right)^2$ 、$h_2^2=\left(\frac{\partial x_1}{\partial q_2}\right)^2+\left(\frac{\partial x_2}{\partial q_2}\right)^2+\left(\frac{\partial x_3}{\partial q_2}\right)^2$ 、$h_3^2=\left(\frac{\partial x_1}{\partial q_3}\right)^2+\left(\frac{\partial x_2}{\partial q_3}\right)^2+\left(\frac{\partial x_3}{\partial q_3}\right)^2$ 。

故得證。