提要 233：曲線座標系統之梯度

曲線座標系統之梯度

若(x_1, x_2, x_3)表卡氏座標系統、(q_1, q_2, q_3)表曲線直角座標系統，則曲線座標系統之梯度∇f 為：

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{u} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{v} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{w} \tag{1}$$

其中 $h_1^2 = \left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2$；

$h_2^2 = \left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2$；

$h_3^2 = \left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2$；

$\mathbf{u}$、$\mathbf{v}$、$\mathbf{w}$ 表曲線直角座標系統(q_1, q_2, q_3)之單位方向向量。

【證明】

由之前弧長元素 ds 的證明得知：

$$(ds)^2 = h_1^2(dq_1)^2 + h_2^2(dq_2)^2 + h_3^2(dq_3)^2 \tag{2}$$

其中 $h_1^2 = \left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2$、$h_2^2 = \left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2$、

$h_3^2 = \left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2$。

若令：

$$d\mathbf{q} = h_1 dq_1 \mathbf{u} + h_2 dq_2 \mathbf{v} + h_3 dq_3 \mathbf{w} \tag{3}$$

則式(2)可改寫為：

$$(ds)^2 = d\mathbf{q}\cdot d\mathbf{q} = (h_1 dq_1\mathbf{u} + h_2 dq_2\mathbf{v} + h_3 dq_3\mathbf{w})\cdot(h_1 dq_1\mathbf{u} + h_2 dq_2\mathbf{v} + h_3 dq_3\mathbf{w}) \tag{4}$$

又弧長元素 ds 亦可表為：

$$(ds)^2 = (ds_1\mathbf{u} + ds_2\mathbf{v} + ds_3\mathbf{w})\cdot(ds_1\mathbf{u} + ds_2\mathbf{v} + ds_3\mathbf{w}) \tag{5}$$

比較式(4)與式(5)可知：

$$h_1 dq_1 = ds_1 \text{、} h_2 dq_2 = ds_2 \text{、} h_3 dq_3 = ds_3 \tag{6}$$

也就是說：

$$\frac{\partial q_1}{\partial s_1} = \frac{1}{h_1} \text{、} \frac{\partial q_2}{\partial s_2} = \frac{1}{h_2} \text{、} \frac{\partial q_3}{\partial s_3} = \frac{1}{h_3} \tag{7}$$

最後再來看純量函數 f 的梯度：

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial s_1}\mathbf{u} + \frac{\partial f}{\partial s_2}\mathbf{v} + \frac{\partial f}{\partial s_3}\mathbf{w} \\
&= \left[\frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial s_1} + \frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial s_1} + \frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial s_1}\right]\mathbf{u} \\
&+ \left[\frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial s_2} + \frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial s_2} + \frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial s_2}\right]\mathbf{v} \\
&+ \left[\frac{\partial f}{\partial q_1}\frac{\partial q_1}{\partial s_3} + \frac{\partial f}{\partial q_2}\frac{\partial q_2}{\partial s_3} + \frac{\partial f}{\partial q_3}\frac{\partial q_3}{\partial s_3}\right]\mathbf{w} \\
&= \left[\frac{\partial f}{\partial q_1}\left(\frac{1}{h_1}\right) + \frac{\partial f}{\partial q_2}(0) + \frac{\partial f}{\partial q_3}(0)\right]\mathbf{u} \\
&+ \left[\frac{\partial f}{\partial q_1}(0) + \frac{\partial f}{\partial q_2}\left(\frac{1}{h_2}\right) + \frac{\partial f}{\partial q_3}(0)\right]\mathbf{v} \\
&+ \left[\frac{\partial f}{\partial q_1}(0) + \frac{\partial f}{\partial q_2}(0) + \frac{\partial f}{\partial q_3}\left(\frac{1}{h_3}\right)\right]\mathbf{w} \\
&= \frac{1}{h_1}\frac{\partial f}{\partial q_1}\mathbf{u} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\mathbf{v} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\mathbf{w}
\end{aligned} \tag{8}$$

故得證。

【附註】

- 圓柱座標系統中函數 f 的梯度為 $\nabla f=\frac{\partial f}{\partial r}\mathbf{u}+\frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{v}+\frac{\partial f}{\partial z}\mathbf{w}$ 。
- 球體座標系統中函數 f 的梯度為 $\nabla f=\frac{\partial f}{\partial r}\mathbf{u}+\frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{v}+\frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{w}$ 。