提要 232：曲線座標系統之弧長

以下幾個單元都要說到曲線座標系統的概念，需要有一點想像力，若讀者不習慣，則只要弄懂圓柱座標(Cylindrical Coordinate)與球體座標(Spherical Coordinate)即可。

曲線座標系統之弧長

若(x_1, x_2, x_3)表卡氏座標系統、(q_1, q_2, q_3)表曲線直角座標系統，則曲線座標系統之弧長ds為：

$$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 = h_1^2(dq_1)^2 + h_2^2(dq_2)^2 + h_3^2(dq_3)^2 \tag{1}$$

其中 $h_1^2 = \left(\dfrac{\partial x_1}{\partial q_1}\right)^2 + \left(\dfrac{\partial x_2}{\partial q_1}\right)^2 + \left(\dfrac{\partial x_3}{\partial q_1}\right)^2$；

$h_2^2 = \left(\dfrac{\partial x_1}{\partial q_2}\right)^2 + \left(\dfrac{\partial x_2}{\partial q_2}\right)^2 + \left(\dfrac{\partial x_3}{\partial q_2}\right)^2$；

$h_3^2 = \left(\dfrac{\partial x_1}{\partial q_3}\right)^2 + \left(\dfrac{\partial x_2}{\partial q_3}\right)^2 + \left(\dfrac{\partial x_3}{\partial q_3}\right)^2$。

【證明】

首先將卡氏座標系統之符號(x, y, z)改寫為(x_1, x_2, x_3)，則弧長元素ds的平方可表為：

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 = \sum_{i=1}^{3}(dx_i)^2 \tag{1}$$

任意之曲線座標系統(q_1, q_2, q_3)一定可以找到它跟卡氏座標系統(x, y, z)的關係，亦即：

$$x_1 = x_1(q_1, q_2, q_3)、x_2 = x_2(q_1, q_2, q_3)、x_3 = x_3(q_1, q_2, q_3) \tag{2}$$

故

$$dx_1 = \frac{\partial x_1}{\partial q_1}dq_1 + \frac{\partial x_1}{\partial q_2}dq_2 + \frac{\partial x_1}{\partial q_3}dq_3 = \sum_{j=1}^{3}\frac{\partial x_1}{\partial q_j}dq_j \tag{3a}$$

$$dx_2 = \frac{\partial x_2}{\partial q_1} dq_1 + \frac{\partial x_2}{\partial q_2} dq_2 + \frac{\partial x_2}{\partial q_3} dq_3 = \sum_{j=1}^{3} \frac{\partial x_2}{\partial q_j} dq_j \tag{3b}$$

$$dx_3 = \frac{\partial x_3}{\partial q_1} dq_1 + \frac{\partial x_3}{\partial q_2} dq_2 + \frac{\partial x_3}{\partial q_3} dq_3 = \sum_{j=1}^{3} \frac{\partial x_3}{\partial q_j} dq_j \tag{3c}$$

所以

$$\begin{aligned}
(ds)^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \\
&= \left(\frac{\partial x_1}{\partial q_1} dq_1 + \frac{\partial x_1}{\partial q_2} dq_2 + \frac{\partial x_1}{\partial q_3} dq_3 \right)^2 \\
&+ \left(\frac{\partial x_2}{\partial q_1} dq_1 + \frac{\partial x_2}{\partial q_2} dq_2 + \frac{\partial x_2}{\partial q_3} dq_3 \right)^2 \\
&+ \left(\frac{\partial x_3}{\partial q_1} dq_1 + \frac{\partial x_3}{\partial q_2} dq_2 + \frac{\partial x_3}{\partial q_3} dq_3 \right)^2 \\
&= \left[\left(\frac{\partial x_1}{\partial q_1} \right)^2 + \left(\frac{\partial x_2}{\partial q_1} \right)^2 + \left(\frac{\partial x_3}{\partial q_1} \right)^2 \right] (dq_1)^2 \\
&+ \left[\left(\frac{\partial x_1}{\partial q_2} \right)^2 + \left(\frac{\partial x_2}{\partial q_2} \right)^2 + \left(\frac{\partial x_3}{\partial q_2} \right)^2 \right] (dq_2)^2 \\
&+ \left[\left(\frac{\partial x_1}{\partial q_3} \right)^2 + \left(\frac{\partial x_2}{\partial q_3} \right)^2 + \left(\frac{\partial x_3}{\partial q_3} \right)^2 \right] (dq_3)^2 \\
&+ 2 \frac{\partial x_1}{\partial q_1} \frac{\partial x_1}{\partial q_2} dq_1 dq_2 + 2 \frac{\partial x_1}{\partial q_1} \frac{\partial x_1}{\partial q_3} dq_1 dq_3 + 2 \frac{\partial x_1}{\partial q_2} \frac{\partial x_1}{\partial q_3} dq_2 dq_3 \\
&+ 2 \frac{\partial x_2}{\partial q_1} \frac{\partial x_2}{\partial q_2} dq_1 dq_2 + 2 \frac{\partial x_2}{\partial q_1} \frac{\partial x_2}{\partial q_3} dq_1 dq_3 + 2 \frac{\partial x_2}{\partial q_2} \frac{\partial x_2}{\partial q_3} dq_2 dq_3 \\
&+ 2 \frac{\partial x_3}{\partial q_1} \frac{\partial x_3}{\partial q_2} dq_1 dq_2 + 2 \frac{\partial x_3}{\partial q_1} \frac{\partial x_3}{\partial q_3} dq_1 dq_3 + 2 \frac{\partial x_3}{\partial q_2} \frac{\partial x_3}{\partial q_3} dq_2 dq_3
\end{aligned}$$

因為(x_1, x_2, x_3)與(q_1, q_2, q_3)均考慮為直角座標系統，故：

$$
\begin{aligned}
&2\frac{\partial x_1}{\partial q_1}\frac{\partial x_1}{\partial q_2}dq_1dq_2+2\frac{\partial x_1}{\partial q_1}\frac{\partial x_1}{\partial q_3}dq_1dq_3+2\frac{\partial x_1}{\partial q_2}\frac{\partial x_1}{\partial q_3}dq_2dq_3\\
&+2\frac{\partial x_2}{\partial q_1}\frac{\partial x_2}{\partial q_2}dq_1dq_2+2\frac{\partial x_2}{\partial q_1}\frac{\partial x_2}{\partial q_3}dq_1dq_3+2\frac{\partial x_2}{\partial q_2}\frac{\partial x_2}{\partial q_3}dq_2dq_3\\
&+2\frac{\partial x_3}{\partial q_1}\frac{\partial x_3}{\partial q_2}dq_1dq_2+2\frac{\partial x_3}{\partial q_1}\frac{\partial x_3}{\partial q_3}dq_1dq_3+2\frac{\partial x_3}{\partial q_2}\frac{\partial x_3}{\partial q_3}dq_2dq=0
\end{aligned}
$$

因此：

$$
\begin{aligned}
(ds)^2=&\left[\left(\frac{\partial x_1}{\partial q_1}\right)^2+\left(\frac{\partial x_2}{\partial q_1}\right)^2+\left(\frac{\partial x_3}{\partial q_1}\right)^2\right](dq_1)^2\\
&+\left[\left(\frac{\partial x_1}{\partial q_2}\right)^2+\left(\frac{\partial x_2}{\partial q_2}\right)^2+\left(\frac{\partial x_3}{\partial q_2}\right)^2\right](dq_2)^2\\
&+\left[\left(\frac{\partial x_1}{\partial q_3}\right)^2+\left(\frac{\partial x_2}{\partial q_3}\right)^2+\left(\frac{\partial x_3}{\partial q_3}\right)^2\right](dq_3)^2\\
&=h_1^2(dq_1)^2+h_2^2(dq_2)^2+h_3^2(dq_3)^2
\end{aligned}
\tag{4}
$$

其中 $h_1^2=\left(\frac{\partial x_1}{\partial q_1}\right)^2+\left(\frac{\partial x_2}{\partial q_1}\right)^2+\left(\frac{\partial x_3}{\partial q_1}\right)^2$ 、 $h_2^2=\left(\frac{\partial x_1}{\partial q_2}\right)^2+\left(\frac{\partial x_2}{\partial q_2}\right)^2+\left(\frac{\partial x_3}{\partial q_2}\right)^2$ 、

$h_3^2=\left(\frac{\partial x_1}{\partial q_3}\right)^2+\left(\frac{\partial x_2}{\partial q_3}\right)^2+\left(\frac{\partial x_3}{\partial q_3}\right)^2$ ，故得證。

範例一

試證明圓柱座標系統(r,θ,z)之弧長ds為：

$$(ds)^2=(dr)^2+r^2(d\theta)^2+(dz)^2$$

【證明】

已知圓柱座標系統(r,θ,z)與卡氏座標系統(x,y,z)的關係為：

$$x=r\cos\theta \text{ 、 } y=r\sin\theta \text{ 、 } z=z$$

若將圓柱座標系統(r,θ,z)之符號改為(q_1,q_2,q_3)、卡氏座標系統(x,y,z)之符號改為(x_1,x_2,x_3)，則

$$x_1=q_1\cos q_2 \text{ 、 } x_2=q_1\sin q_2 \text{ 、 } x_3=q_3$$

則

$$dx_1=\frac{\partial x_1}{\partial q_1}dq_1+\frac{\partial x_1}{\partial q_2}dq_2+\frac{\partial x_1}{\partial q_3}dq_3=\cos q_2dq_1-q_1\sin q_2dq_2$$
$$dx_2=\frac{\partial x_2}{\partial q_1}dq_1+\frac{\partial x_2}{\partial q_2}dq_2+\frac{\partial x_2}{\partial q_3}dq_3=\sin q_2dq_1+q_1\cos q_2dq_2$$
$$dx_3=\frac{\partial x_3}{\partial q_1}dq_1+\frac{\partial x_3}{\partial q_2}dq_2+\frac{\partial x_3}{\partial q_3}dq_3=dq_3$$

所以

$$\begin{aligned}(ds)^2&=(dx_1)^2+(dx_2)^2+(dx_3)^2\\&=(\cos q_2dq_1-q_1\sin q_2dq_2)^2\\&\quad+(\sin q_2dq_1+q_1\cos q_2dq_2)^2\\&\quad+(dq_3)^2\\&=(dq_1)^2+q_1^2(dq_2)^2+(dq_3)^2\end{aligned}$$

也就是說：

$$(ds)^2=(dr)^2+r^2(d\theta)^2+(dz)^2$$

故得證。

【附註】

1. 球體座標(r,θ,ϕ)之弧長元素ds的平方為$(ds)^2=(dr)^2+r^2\sin^2\phi(d\theta)^2+r^2(d\phi)^2$。