提要 141：*Hankel* 轉換之應用

筆者的研究工作與 *Hankel* 轉換之應用密切相關，因此就舉兩例加以解釋，說明如下。

多孔介質熱彈性力學的無限域及半無限域穩態基本解

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摘　　要

本文模擬線彈性飽和多孔介質爲均質均向性之無限域及半無限域，引用積分轉換方法，分別研討出飽和多孔介質受任意方向點作用力源、點熱源與點補注水源作用時之穩態基本解。所研討出之三維基本解爲以邊界元素法解析多孔介質的熱彈性力學問題之基礎。

關鍵詞：多孔介質，無限域，半無限域，基本解。

緒　　論

多孔介質熱彈性力學理論(thermoporoelasticity)係由壓密(consolidation)理論開始發展，而合理的多孔介質三維壓密理論則由Biot[1，2]首先研討出，許多探討多孔介質彈力問題的研究，均是引用此一理論建立數學模式。爲探討多孔介質在受熱狀態下之力學行爲變化，在後續的研究中，Schiffman[3]曾引用Fourier定律，以熱傳導的觀念說明多孔介質中之熱量傳輸現象，建立多孔介質之熱彈性力學理論。相關之多孔介質熱彈性力學理論模式尙有許多不同的型式，有些是定義新的力學參數[4]建立理論模式，有些則是增加考慮熱對流效應的影響[5]，但均是以Biot[1，2]所建立之多孔介質彈性力學理論模式爲基礎。

近來，以邊界元素法(boundary element method)解析多孔介質之熱彈力問題是一個重要的研究方向，而以邊界元素法解析物理問題之重要關鍵爲需獲得適用之基本解(fundamental solution)。以往相關問題之基本解的研究，多不考慮多孔介質中之熱量及溫度場的變化[6-12]，而有考慮溫度變化效應的多孔介質無限域(full-space)基本解，首先是由Smith與Booker[13]提出的。然而文獻[13]並未考慮半無限域(half-space)之基本解，若半無限域之適用基本解可以獲得，將有助於以邊界元素法解析多孔介質之半無限域熱彈性力學問題。基於此，本文除探討無限域模式之基本解外，亦將進一步研討多孔介質之半無限域基本解。

數學模式中，模擬飽和多孔介質爲均質均向性之線彈性體，考慮孔隙流體的流動遵守質量平衡定理與Darcy定律，熱量之擴散傳輸服從熱平衡方程式與

Fourier定律。所研討出之基本解，係考慮介質內部受到定速率補注之點熱源、點水源、以及任意方向之點作用力源作用時所引致之力學行爲變化。數學模式之解析係採用積分轉換方法(integral transform)，所研討出之基本解爲穩態的閉合解。

本文共研討出五種多孔介質熱彈性力學的穩態基本解。其中四種基本解爲半無限域之穩態基本解，這四種基本解的主要差異爲所考慮之半平面邊界的滲流及熱流邊界條件並不完全相同；第五種則是無限域之穩態基本解。可應用這些基本解與已推導完成之邊界積分方程式[14]，撰寫邊界元素計算程式，解析牽涉熱流場變化所引致的多孔介質力學問題。

數學模式

數學模式中之多孔介質模擬分成兩種情況，分別爲無限域及半無限域多孔介質。爲方便解析及說明，本文考慮無限域及半無限域多孔介質內，均於 $(r,\theta,z)=(0,0,h)$ 位置承受任意方向之點作用力源 (F_r,F_θ,F_z) 作用，且單位時間內有 Q 體積之水源和 W 熱量之熱源持續補注於多孔介質中之同一點，如圖1與圖2所示。點源作用於其他位置所引致之基本解，僅需將所研討出之解作適當之座標平移即可。

若以介質位移 (u_r,u_θ,u_z)、超額孔隙水壓力 p (壓力爲正)、及介質溫度變化量 ϑ 等爲基本變數，模擬多孔介質爲均質均向性之飽和線彈性體，考慮孔隙流體的流動遵守質量平衡定理與Darcy定律，熱量之擴散傳輸服從熱平衡方程式與Fourier定律，並讓座標 z 軸通過作用源點，則考慮固體介質與孔隙水爲可壓縮之理論模式的基本控制方程式可以圓柱座標 (r,θ,z) 表示爲[14]：

$$G\nabla^2 u_r + G(2\eta-1)\frac{\partial\varepsilon}{\partial r} - \frac{G}{r}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)$$

$$-\frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}\frac{\partial p}{\partial r} - \frac{2G\alpha_s(1+\nu)}{1-2\nu}\frac{\partial\vartheta}{\partial r}$$

$$+\frac{F_r}{r}\delta(r)\delta(\theta)\delta(z-h)=0\text{，}\quad (1a)$$

$$G\nabla^2 u_\theta + G(2\eta-1)\frac{1}{r}\frac{\partial\varepsilon}{\partial\theta} + \frac{G}{r}\left(\frac{2}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)$$

$$-\frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}\frac{1}{r}\frac{\partial p}{\partial\theta} - \frac{2G\alpha_s(1+\nu)}{1-2\nu}\frac{1}{r}\frac{\partial\vartheta}{\partial\theta}$$

$$+\frac{F_\theta}{r}\delta(r)\delta(\theta)\delta(z-h)=0\text{，}\quad (1b)$$

$$G\nabla^2 u_z + G(2\eta-1)\frac{\partial\varepsilon}{\partial z} - \frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}\frac{\partial p}{\partial z}$$

$$-\frac{2G\alpha_s(1+\nu)}{1-2\nu}\frac{\partial\vartheta}{\partial z} + \frac{F_z}{r}\delta(r)\delta(\theta)\delta(z-h)=0\text{，}\quad (1c)$$

$$\frac{k}{\gamma_w}\nabla^2 p + \frac{Q}{r}\delta(r)\delta(\theta)\delta(z-h)=0\text{，}\quad (1d)$$

$$\lambda_t\nabla^2\vartheta + \frac{W}{r}\delta(r)\delta(\theta)\delta(z-h)=0\text{，}\quad (1e)$$

式中 $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$；$\delta(x)$=Dirac-delta函數；$\varepsilon=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}$，是多孔介質之體積應變量；$B$ = Skempton孔隙水壓力參數；G=多孔介質之剪力係數(shear modulus)；k=多孔介質之滲透係數(permeability)；ν_u=不排水情況(undrained)下所測得之多孔介質柏松比(Poisson's ratio)；$\eta=\frac{1-\nu}{1-2\nu}$，$\nu$=排水情況(drained)下所測得之多孔介質柏松比；$\frac{2G\alpha_s(1+\nu)}{1-2\nu}=(2G+3\lambda)\alpha_s$，$\lambda$ 係多孔介質之Lame常數；α_s=固體介質之線性熱膨脹係數(linear thermal expansion coefficient)；γ_w=孔隙水之單位重；λ_t=多孔介質之熱傳導係數。

數學模式中之邊界條件有三類，分別爲力學邊界條件(mechanical boundary condition)、滲流邊界條件(hydraulic boundary condition)、與熱流邊界條件(thermal boundary condition)。若考慮多孔介質爲無限

域，則其在深遠邊界上$(z \to \pm\infty)$之各種物理變化量可考慮爲不受作用源的影響，即：

$$\lim_{z \to \infty}\{u_r, u_\theta, u_z, p, \vartheta\} \to \{0,0,0,0,0\}， \tag{2a}$$

$$\lim_{z \to -\infty}\{u_r, u_\theta, u_z, p, \vartheta\} \to \{0,0,0,0,0\}。 \tag{2b}$$

數學模式中若考慮多孔介質爲半無限域情況，則其無限深遠處$(z \to \infty)$之邊界條件與式(2a)相同；然而其在$z=0$之半平面邊界上力學邊界條件係考慮爲有效應力(effective stress) σ_{ij} 沒有變化，即：

$$\sigma_{rz}(r,\theta,0) = G\left[\frac{\partial u_r(r,\theta,0)}{\partial z} + \frac{\partial u_z(r,\theta,0)}{\partial r}\right] = 0， \tag{3a}$$

$$\sigma_{\theta z}(r,\theta,0) = G\left[\frac{\partial u_\theta(r,\theta,0)}{\partial z} + \frac{1}{r}\frac{\partial u_z(r,\theta,0)}{\partial \theta}\right] = 0， \tag{3b}$$

$$\sigma_{zz}(r,\theta,0) = G\left\{2(\eta-1)\left[\frac{\partial u_r(r,\theta,0)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta(r,\theta,0)}{\partial \theta} + \frac{u_r(r,\theta,0)}{r}\right] + 2\eta\frac{\partial u_z(r,\theta,0)}{\partial z} - \frac{2\alpha_s(1+\nu)}{1-2\nu}\vartheta(r,\theta,0)\right\} = 0， \tag{3c}$$

式中有效應力σ_{ij}與總應力(total stress) τ_{ij}的關係爲：

$$\tau_{ij} = \sigma_{ij} - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} p\delta_{ij}， \tag{4}$$

其中δ_{ij}爲Kronecker delta函數，當$i=j$時，$\delta_{ij}=1$，當$i \neq j$時，$\delta_{ij}=0$。

半平面邊界上$(z=0)$之滲流邊界條件，可分別考慮爲完全透水情況或完全不透水情況，即：

$$p(r,\theta,0) = 0， \tag{5}$$

$$\frac{\partial p(r,\theta,0)}{\partial z} = 0。 \tag{6}$$

$z=0$之半平面邊界的熱流邊界條件，可分別考慮爲保持恆溫狀態或隔熱狀態，即半平面邊界不產生溫度變化或無熱量變化：

$$\vartheta(r,\theta,0) = 0， \tag{7}$$

$$\frac{\partial \vartheta(r,\theta,0)}{\partial z} = 0。 \tag{8}$$

本文將分別研討上述各種不同邊界條件下，多孔介質熱彈性力學的無限域及半無限域穩態基本解。

積分轉換解析

數學模式之解析是採用Fourier指數積分轉換與Hankel積分轉換方法。首先對數學模式中之變數θ作Fourier指數積分轉換，則式(1a)至式(1e)變換爲：

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + (2\eta-1)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) - \frac{1+\alpha^2}{r^2}\right]U_r + i\alpha\left[(2\eta-1)\left(\frac{1}{r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{2}{r^2}\right]U_\theta + (2\eta-1)\frac{\partial^2 U_z}{\partial r \partial z}$$
$$- \frac{3(\nu_u - \nu)}{GB(1-2\nu)(1+\nu_u)}\frac{\partial P}{\partial r} - \frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{\partial \Theta}{\partial r} + \frac{1}{r}\frac{F_r}{G}\delta(r)\delta(z-h) = 0， \tag{9a}$$

$$-i\alpha\left[(2\eta-1)\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\right) + \frac{2}{r^2}\right]U_r + \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - (2\eta-1)\frac{\alpha^2}{r^2} - \frac{1+\alpha^2}{r^2}\right]U_\theta$$
$$- i\alpha(2\eta-1)\frac{1}{r}\frac{\partial U_z}{\partial z} + i\alpha\frac{3(\nu_u - \nu)}{GB(1-2\nu)(1+\nu_u)}\frac{P}{r} + i\alpha\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{\Theta}{r} + \frac{1}{r}\frac{F_\theta}{G}\delta(r)\delta(z-h) = 0， \tag{9b}$$

$$(2\eta-1)\left(\frac{\partial^2}{\partial r \partial z} + \frac{1}{r}\frac{\partial}{\partial z}\right)U_r - i\alpha(2\eta-1)\frac{1}{r}\frac{\partial U_\theta}{\partial z}$$

$$+\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\alpha^2}{r^2}+2\eta\frac{\partial^2}{\partial z^2}\right)U_z-\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{\partial P}{\partial z}$$

$$-\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{\partial\Theta}{\partial z}+\frac{1}{r}\frac{F_z}{G}\delta(r)\delta(z-h)=0\text{ ,}\qquad(9c)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\alpha^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)P+\frac{1}{r}\frac{Q\gamma_w}{k}\delta(r)\delta(z-h)=0\text{ ,}\qquad(9d)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\alpha^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)\Theta+\frac{1}{r}\frac{W}{\lambda_t}\delta(r)\delta(z-h)=0\text{ ,}\qquad(9e)$$

式中 α 為 Fourier 指數積分轉換參數；$(U_r,U_\theta,U_z,P,\Theta)$ 為 $(u_r,u_\theta,u_z,p,\vartheta)$ 之Fourier指數積分轉換函數：

$$\{U_r(r,z;\alpha),U_\theta(r,z;\alpha),U_z(r,z;\alpha),P(r,z;\alpha),\Theta(r,z;\alpha)\}$$

$$=\int_0^{2\pi}\{u_r,u_\theta,u_z,p,\vartheta\}\exp(i\alpha\theta)d\theta\text{ ;}\qquad(10)$$

其反轉換係定義為：

$$\{u_r(r,\theta,z),u_\theta(r,\theta,z),u_z(r,\theta,z),p(r,\theta,z),\vartheta(r,\theta,z)\}$$

$$=\frac{1}{2\pi}\sum_{\alpha=-\infty}^{\infty}\{U_r,U_\theta,U_z,P,\Theta\}\exp(-i\alpha\theta)\text{ 。}\qquad(11)$$

式(9a)至式(9e)甚為複雜，不易直接以Hankel積分轉換方法處理，需作適當之變數變換。令：

$$\Phi=U_r+iU_\theta\text{ ，}\Psi=U_r-iU_\theta\text{ ，}\qquad(12)$$

則式(9a)至式(9e)可以 Φ 、Ψ 、U_z 、P 、Θ 表示。將代入新函數 Φ 與 Ψ 後之式(9a)加式(9b)暨式(9a)減式(9b)，再分別對其中之 r 變數作 $\alpha-1$ 階與 $\alpha+1$ 階之Hankel積分轉換，同時式(9c)、式(9d)與式(9e)亦分別對其中之 r 變數作 α 階之Hankel積分轉換，則可將聯立偏微分方程式化簡為聯立常微分方程式如下：

$$\left[2\left(\frac{d^2}{dz^2}-\xi^2\right)-(2\eta-1)\xi^2\right]\tilde{\Phi}+(2\eta-1)\xi^2\tilde{\Psi}$$

$$+2(2\eta-1)\xi\frac{d\tilde{U}_z}{dz}-\frac{6(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\xi\tilde{P}$$

$$-\frac{4\alpha_s(1+\nu)}{1-2\nu}\xi\tilde{\Theta}+\frac{2(F_r+iF_\theta)}{G}\delta(z-h)\delta_{1\alpha}=0\text{ ，}\qquad(13a)$$

$$(2\eta-1)\xi^2\tilde{\Phi}+\left[2\left(\frac{d^2}{dz^2}-\xi^2\right)-(2\eta-1)\xi^2\right]\tilde{\Psi}$$

$$-2(2\eta-1)\xi\frac{d\tilde{U}_z}{dz}+\frac{6(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\xi\tilde{P}$$

$$+\frac{4\alpha_s(1+\nu)}{1-2\nu}\xi\tilde{\Theta}+\frac{2(F_r-iF_\theta)}{G}\delta(z-h)\delta_{-1\alpha}=0\text{ ，}\qquad(13b)$$

$$-(2\eta-1)\xi\frac{d\tilde{\Phi}}{dz}+(2\eta-1)\xi\frac{d\tilde{\Psi}}{dz}$$

$$+2\left(2\eta\frac{d^2}{dz^2}-\xi^2\right)\tilde{U}_z-\frac{6(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{d\tilde{P}}{dz}$$

$$-\frac{4\alpha_s(1+\nu)}{1-2\nu}\frac{d\tilde{\Theta}}{dz}+\frac{2F_z}{G}\delta(z-h)\delta_{0\alpha}=0\text{ ，}\qquad(13c)$$

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\tilde{P}+\frac{Q\gamma_w}{k}\delta(z-h)\delta_{0\alpha}=0\text{ ，}\qquad(13d)$$

$$\left(\frac{d^2}{dz^2}-\xi^2\right)\tilde{\Theta}+\frac{W}{\lambda_t}\delta(z-h)\delta_{0\alpha}=0\text{ ，}\qquad(13e)$$

式中 ξ 為Hankel積分轉換參數；$\tilde{\Phi}$ 、$\tilde{\Psi}$ 、$\tilde{U}_z$ 、$\tilde{P}$ 、$\tilde{\Theta}$ 分別為 Φ 、Ψ 、U_z 、P 、Θ 之Hankel積分轉換函數：

$$\tilde{\Phi}(z;\alpha,\xi)=\int_0^\infty r\,\Phi(r,z;\alpha)\,J_{\alpha-1}(\xi r)dr\text{ ，}\qquad(14a)$$

$$\tilde{\Psi}(z;\alpha,\xi)=\int_0^\infty r\Psi(r,z;\alpha)\,J_{\alpha+1}(\xi r)dr\text{ ，}\qquad(14b)$$

$$\tilde{U}_z(z;\alpha,\xi)=\int_0^\infty r\,U_z(r,z;\alpha)\,J_\alpha(\xi r)dr\text{ ，}\qquad(14c)$$

$$\tilde{P}(z;\alpha,\xi)=\int_0^\infty r\,P(r,z;\alpha)\,J_\alpha(\xi r)dr\text{ ，}\qquad(14d)$$

$$\tilde{\Theta}(z;\alpha,\xi)=\int_0^\infty r\,\Theta(r,z;\alpha)\,J_\alpha(\xi r)dr\text{ ；}\qquad(14e)$$

其反轉換係定義爲：

$$\Phi(r,z;\alpha)=\int_0^\infty \xi\,\tilde{\Phi}(z;\alpha,\xi)\,J_{\alpha-1}(\xi r)\,d\xi\text{，}\qquad(15a)$$

$$\Psi(r,z;\alpha)=\int_0^\infty \xi\,\tilde{\Psi}(z;\alpha,\xi)\,J_{\alpha+1}(\xi r)\,d\xi\text{，}\qquad(15b)$$

$$U_z(r,z;\alpha)=\int_0^\infty \xi\,\tilde{U}_z(z;\alpha,\xi)\,J_{\alpha}(\xi r)\,d\xi\text{，}\qquad(15c)$$

$$P(r,z;\alpha)=\int_0^\infty \xi\,\tilde{P}(z;\alpha,\xi)\,J_{\alpha}(\xi r)\,d\xi\text{，}\qquad(15d)$$

$$\Theta(r,z;\alpha)=\int_0^\infty \xi\,\tilde{\Theta}(z;\alpha,\xi)\,J_{\alpha}(\xi r)\,d\xi\text{。}\qquad(15e)$$

式(13a)至式(13e)之通解(general solution)包含齊性解(homogeneous solution)與非齊性解兩部分，其解析過程相當繁瑣，但有標準方法可循，經仔細研討後，可得 $\tilde{\Phi}$ 、 $\tilde{\Psi}$ 、 $\tilde{U}_z$ 、 $\tilde{P}$ 、 $\tilde{\Theta}$ 之通解如下：

$$\tilde{\Phi}(z;\alpha,\xi)=(C_1+C_2z)\exp(\xi z)+(C_3+C_4z)\exp(-\xi z)$$
$$+\frac{F_r+iF_\theta}{G}\left[\frac{1}{2\xi}-\frac{2\eta-1}{16\eta}\left(\frac{1}{\xi}+|z-h|\right)\right]\exp(-\xi|z-h|)\delta_{1\alpha}$$
$$+\frac{F_r-iF_\theta}{G}\left[\frac{2\eta-1}{16\eta}\left(\frac{1}{\xi}+|z-h|\right)\right]\exp(-\xi|z-h|)\delta_{-1\alpha}$$
$$+\frac{F_z}{G}\left[-\frac{2\eta-1}{8\eta}(z-h)\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{Q\gamma_w}{k}\left[-\frac{1}{8\eta}\left(\frac{1}{\xi^2}+\frac{|z-h|}{\xi}\right)\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{W}{\lambda_t}\left[-\frac{1}{8\eta}\left(\frac{1}{\xi^2}+\frac{|z-h|}{\xi}\right)\right]\exp(-\xi|z-h|)\delta_{0\alpha}\text{，}\qquad(16a)$$

$$\tilde{\Psi}(z;\alpha,\xi)=(C_5-C_2z)\exp(\xi z)+(C_6-C_4z)\exp(-\xi z)$$
$$+\frac{F_r+iF_\theta}{G}\left[\frac{2\eta-1}{16\eta}\left(\frac{1}{\xi}+|z-h|\right)\right]\exp(-\xi|z-h|)\delta_{1\alpha}$$
$$+\frac{F_r-iF_\theta}{G}\left[\frac{1}{2\xi}-\frac{2\eta-1}{16\eta}\left(\frac{1}{\xi}+|z-h|\right)\right]\exp(-\xi|z-h|)\delta_{-1\alpha}$$
$$+\frac{F_z}{G}\left[\frac{2\eta-1}{8\eta}(z-h)\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{Q\gamma_w}{k}\left[\frac{1}{8\eta}\left(\frac{1}{\xi^2}+\frac{|z-h|}{\xi}\right)\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{W}{\lambda_t}\left[\frac{1}{8\eta}\left(\frac{1}{\xi^2}+\frac{|z-h|}{\xi}\right)\right]\exp(-\xi|z-h|)\delta_{0\alpha}\text{，}\qquad(16b)$$

$$\tilde{U}_z(z;\alpha,\xi)=\left[\frac{1}{2}C_1-\frac{2\eta+1}{2\eta-1}\frac{1}{\xi}C_2-\frac{1}{2}C_5\right.$$
$$+\frac{1}{2\eta-1}\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{1}{\xi}C_7$$
$$\left.+\frac{1}{2\eta-1}\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{1}{\xi}C_9+C_2z\right]\exp(\xi z)$$
$$+\left[-\frac{1}{2}C_3-\frac{2\eta+1}{2\eta-1}\frac{1}{\xi}C_4+\frac{1}{2}C_6\right.$$
$$-\frac{1}{2\eta-1}\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{1}{\xi}C_8$$
$$\left.-\frac{1}{2\eta-1}\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{1}{\xi}C_{10}-C_4z\right]\exp(-\xi z)$$
$$+\frac{F_r+iF_\theta}{G}\left[\frac{2\eta-1}{16\eta}(z-h)\right]\exp(-|z-h|)\delta_{1\alpha}$$
$$+\frac{F_r-iF_\theta}{G}\left[-\frac{2\eta-1}{16\eta}(z-h)\right]\exp(-|z-h|)\delta_{-1\alpha}$$
$$+\frac{F_z}{G}\left[\frac{2\eta+1}{8\eta}\frac{1}{\xi}+\frac{2\eta-1}{8\eta}|z-h|\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{3(\nu_u-\nu)}{GB(1-2\nu)(1+\nu_u)}\frac{Q\gamma_w}{k}\left[\frac{1}{8\eta}\frac{z-h}{\xi}\right]\exp(-\xi|z-h|)\delta_{0\alpha}$$
$$+\frac{2\alpha_s(1+\nu)}{1-2\nu}\frac{W}{\lambda_t}\left[\frac{1}{8\eta}\frac{z-h}{\xi}\right]\exp(-\xi|z-h|)\delta_{0\alpha}\text{，}\qquad(16c)$$

$$\tilde{P}(z;\alpha,\xi)=C_7\exp(\xi z)+C_8\exp(-\xi z)$$
$$+\frac{Q\gamma_w}{2k}\frac{1}{\xi}\exp(-\xi|z-h|)\delta_{0\alpha}\text{，}\qquad(16d)$$

$$\tilde{\Theta}(z;\alpha,\xi)=C_9\exp(\xi z)+C_{10}\exp(-\xi z)$$

$$+\frac{W}{2\lambda_t}\frac{1}{\xi}\exp\left(-\xi\left|z-h\right|\right)\delta_{0\alpha}\text{ ，} \tag{16e}$$

式中 $C_i\left(i=1,\cdots,10\right)$ 爲待定係數，其值可由數學模式中所欲滿足之邊界條件得出。本文所考慮之多孔介質邊界條件，分別爲1、無限域中之無限深遠 $\left(z\to\pm\infty\right)$ 邊界條件；2、半無限域中之半平面 $\left(z=0\right)$ 邊界條件與無限深遠 $\left(z\to\infty\right)$ 邊界條件。

半無限域中之無限深遠 $\left(z\to\infty\right)$ 邊界條件係考慮各物理量均不受點源的影響，如式(2a)所示，至於地表半平面 $\left(z=0\right)$ 之邊界條件則可區分爲以下四種情況，即：1、半平面邊界 $\left(z=0\right)$ 上無應力變化、完全透水、及保持恆溫狀態，如式(3a)至式(3c)、式(5)、與式(7)所示；2、半平面邊界 $\left(z=0\right)$ 上無應力變化、完全不透水、及保持恆溫狀態，如式(3a)至式(3c)、式(6)、與式(7)所示；3、半平面邊界 $\left(z=0\right)$ 上無應力變化、完全透水、及保持隔熱狀態，如式(3a)至式(3c)、式(5)、與式(8)所示；4、半平面邊界 $\left(z=0\right)$ 上無應力變化、完全不透水、及保持隔熱狀態，如式(3a)至式(3c)、式(6)、與式(8)所示。

以上所述不同種類之邊界條件亦需作適當之積分轉換，方能研討出式(16a)至式(16e)所示通解中之積分轉換域待定係數 $C_i\left(i=1,\cdots,10\right)$ ，各項係數 C_i 均與參數 α 及 ξ 有關。應用 $\varPhi=U_r+iU_\theta$、$\varPsi=U_r-iU_\theta$ 之關係式，以及式(10)之Fourier指數積分轉換公式、與式(14a)至式(14e)所示之Hankel積分轉換公式，則無限域中無限深遠處 $\left(z\to\pm\infty\right)$ 之邊界條件可表爲：

$$\lim_{z\to\infty}\left\{\tilde{\varPhi},\tilde{\varPsi},\tilde{U}_z,\tilde{P},\tilde{\varTheta}\right\}\to\left\{0,0,0,0,0\right\}\text{ ，} \tag{17a}$$

$$\lim_{z\to-\infty}\left\{\tilde{\varPhi},\tilde{\varPsi},\tilde{U}_z,\tilde{P},\tilde{\varTheta}\right\}\to\left\{0,0,0,0,0\right\}\text{ 。} \tag{17b}$$

同理，半無限域中之無限深遠 $\left(z\to\infty\right)$ 邊界條件可表爲式(17a)之型式。而半無限域中之半平面邊界 $\left(z=0\right)$ 若考慮其無應力變化，則：

$$\tilde{F}_1\left(0;\alpha,\xi\right)=G\left[\frac{d\tilde{\varPhi}\left(0;\alpha,\xi\right)}{dz}+\xi\tilde{U}_z\left(0;\alpha,\xi\right)\right]=0\text{ ，} \tag{18a}$$

$$\tilde{F}_2\left(0;\alpha,\xi\right)=G\left[\frac{d\tilde{\varPsi}\left(0;\alpha,\xi\right)}{dz}-\xi\tilde{U}_z\left(0;\alpha,\xi\right)\right]=0\text{ ，} \tag{18b}$$

$$\tilde{S}_{zz}\left(0;\alpha,\xi\right)=G\left[\left(\eta-1\right)\xi\left(\tilde{\varPsi}-\tilde{\varPhi}\right)+2\eta\frac{d\tilde{U}_z}{dz}-\frac{3\left(\nu_u-\nu\right)}{GB\left(1-2\nu\right)\left(1+\nu_u\right)}\tilde{P}-\frac{2\alpha_s\left(1+\nu\right)}{1-2\nu}\tilde{\varTheta}\right]_{z=0}=0\text{ ，} \tag{18c}$$

其中 $\tilde{\varPhi}$ 、$\tilde{\varPsi}$ 、$\tilde{U}_z$ 、$\tilde{P}$ 、$\tilde{\varTheta}$ 如式(14a)至式(14e)所示，而 $\tilde{F}_1$ 、$\tilde{F}_2$ 、$\tilde{S}_{zz}$ 係定義爲：

$$\tilde{F}_1=\int_0^\infty r\left[\int_0^{2\pi}\left(\sigma_{rz}+i\sigma_{\theta z}\right)\exp\left(i\alpha\theta\right)d\theta\right]J_{\alpha-1}\left(\xi r\right)dr\text{ ，} \tag{19a}$$

$$\tilde{F}_2=\int_0^\infty r\left[\int_0^{2\pi}\left(\sigma_{rz}-i\sigma_{\theta z}\right)\exp\left(i\alpha\theta\right)d\theta\right]J_{\alpha+1}\left(\xi r\right)dr\text{ ，} \tag{19b}$$

$$\tilde{S}_{zz}=\int_0^\infty r\left[\int_0^{2\pi}\sigma_{zz}\exp\left(i\alpha\theta\right)d\theta\right]J_{\alpha}\left(\xi r\right)dr\text{ 。} \tag{19c}$$

若考慮半平面邊界 $\left(z=0\right)$ 爲完全透水情況，則式(5)之滲流邊界條件中之變數 θ 與變數 r 分別經由Fourier指數積分轉換與Hankel積分轉換後，積分轉換域之完全透水邊界條件可表爲：

$$\tilde{P}\left(0;\alpha,\xi\right)=0\text{ ；} \tag{20}$$

若考慮半平面邊界 $\left(z=0\right)$ 爲完全不透水，則滲流邊界條件式(6)中之變數 θ 與變數 r 分別經由Fourier指數積分轉換與Hankel積分轉換後，其積分轉換域之邊界條件可表爲：

$$\frac{d\tilde{P}\left(0;\alpha,\xi\right)}{dz}=0\text{ 。} \tag{21}$$

與熱流相關之半平面邊界 $\left(z=0\right)$ 若考慮爲恆溫現象，如式(7)所示，則其中之變數 θ 與變數 r 分別作

Fourier指數積分轉換與Hankel積分轉換後，其積分轉換域之邊界條件爲：

$$\widetilde{\Theta}(0;\alpha,\xi)=0 \text{；} \tag{22}$$

與熱流相關之半平面邊界 $(z=0)$ 若考慮爲隔熱情況，如式(8)所示，則其中之變數 θ 與變數 r 分別作Fourier指數積分轉換與Hankel積分轉換後，此積分轉換域邊界條件爲：

$$\frac{d\widetilde{\Theta}(0;\alpha,\xi)}{dz}=0 \text{。} \tag{23}$$

將式(16a)至式(16e)所示之積分轉換域 $(z;\alpha,\xi)$ 通解代入以上所示之力學、滲流與熱流等邊界條件，即可研討出滿足積分轉換域邊界條件之積分轉換域基本解。所探討出之積分轉換域基本解再引用式(15a)至式(15e)之Hankel反轉換公式、 $U_r=(\Phi+\Psi)/2$ 與 $U_\theta=(\Phi-\Psi)/2i$ 之關係式、以及式(11)所示之Fourier指數反轉換公式，再配合相關之積分公式[15]，即可研討出實數域之基本解。

無限域及半無限域之基本解

1、無限域基本解

經仔細研討後知，無限域之基本解可表爲：

$$u_r(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)f_1(r,z)+F_zf_2(r,z)$$
$$+Qf_3(r,z)+Wf_4(r,z) \text{，} \tag{24a}$$

$$u_\theta(r,\theta,z)=(F_\theta\cos\theta-F_r\sin\theta)f_5(r,z) \text{，} \tag{24b}$$

$$u_z(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)f_6(r,z)+F_zf_7(r,z)$$
$$+Qf_8(r,z)+Wf_9(r,z) \text{，} \tag{24c}$$

$$p(r,z)=\frac{Q\gamma_w}{4\pi k}\frac{1}{R_a} \text{，} \tag{24d}$$

$$\vartheta(r,z)=\frac{W}{4\pi\lambda_t}\frac{1}{R_a} \text{，} \tag{24e}$$

式中函數 $f_i(r,z)(i=1,\cdots,9)$ 與 R_a 分別定義爲：

$$f_1(r,z)=\frac{1}{16\pi\eta G}\left[(2\eta+1)\frac{1}{R_a}+(2\eta-1)\frac{r^2}{R_a^3}\right] \text{，} \tag{25a}$$

$$f_2(r,z)=\frac{2\eta-1}{16\pi\eta G}\frac{r(z-h)}{R_a^3} \text{，} \tag{25b}$$

$$f_3(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\frac{r}{R_a} \text{，} \tag{25c}$$

$$f_4(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\frac{r}{R_a} \text{，} \tag{25d}$$

$$f_5(r,z)=\frac{2\eta+1}{16\pi\eta G}\frac{1}{R_a} \text{，} \tag{25e}$$

$$f_6(r,z)=\frac{2\eta-1}{16\pi\eta G}\frac{r(z-h)}{R_a^3} \text{，} \tag{25f}$$

$$f_7(r,z)=\frac{1}{16\pi\eta G}\left[(2\eta+1)\frac{1}{R_a}+(2\eta-1)\frac{(z-h)^2}{R_a^3}\right] \text{，} \tag{25g}$$

$$f_8(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\frac{z-h}{R_a} \text{，} \tag{25h}$$

$$f_9(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\frac{z-h}{R_a} \text{，} \tag{25i}$$

$$R_a=\sqrt{r^2+(z-h)^2} \text{。} \tag{25j}$$

利用文獻[13]可檢驗出所推導出之無限域基本解是正確的。

2、半無限域基本解

半無限域基本解之型態係隨所滿足之半無限域地表邊界的滲流及熱流條件不同而有所差異，分別列示如下。若考慮半平面邊界無應力變化、完全透水、且保持恆溫狀態，則其半無限域基本解爲：

$$u_r(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_1(r,z)+F_zg_2(r,z)$$
$$+Qg_3(r,z)+Wg_4(r,z)\text{ ，}\quad(26a)$$
$$u_\theta(r,\theta,z)=(F_\theta\cos\theta-F_r\sin\theta)g_5(r,z)\text{ ，}\quad(26b)$$
$$u_z(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_6(r,z)+F_zg_7(r,z)$$
$$+Qg_8(r,z)+Wg_9(r,z)\text{ ，}\quad(26c)$$
$$p(r,z)=\frac{Q\gamma_w}{4\pi k}\left(\frac{1}{R_a}-\frac{1}{R_b}\right)\text{ ，}\quad(26d)$$
$$\vartheta(r,z)=\frac{W}{4\pi\lambda_t}\left(\frac{1}{R_a}-\frac{1}{R_b}\right)\text{ 。}\quad(26e)$$

若考慮半平面邊界無應力變化、完全不透水、且保持恆溫狀態，則其半無限域基本解為：

$$u_r(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_1(r,z)+F_zg_2(r,z)$$
$$+Qg_3^*(r,z)+Wg_4(r,z)\text{ ，}\quad(27a)$$
$$u_\theta(r,\theta,z)=(F_\theta\cos\theta-F_r\sin\theta)g_5(r,z)\text{ ，}\quad(27b)$$
$$u_z(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_6(r,z)+F_zg_7(r,z)$$
$$+Qg_8^*(r,z)+Wg_9(r,z)\text{ ，}\quad(27c)$$
$$p(r,z)=\frac{Q\gamma_w}{4\pi k}\left(\frac{1}{R_a}+\frac{1}{R_b}\right)\text{ ，}\quad(27d)$$
$$\vartheta(r,z)=\frac{W}{4\pi\lambda_t}\left(\frac{1}{R_a}-\frac{1}{R_b}\right)\text{ 。}\quad(27e)$$

若考慮半平面邊界無應力變化、完全透水、且保持隔熱狀態，則所研討出之半無限域基本解為：

$$u_r(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_1(r,z)+F_zg_2(r,z)$$
$$+Qg_3(r,z)+Wg_4^*(r,z)\text{ ，}\quad(28a)$$
$$u_\theta(r,\theta,z)=(F_\theta\cos\theta-F_r\sin\theta)g_5(r,z)\text{ ，}\quad(28b)$$
$$u_z(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_6(r,z)+F_zg_7(r,z)$$
$$+Qg_8(r,z)+Wg_9^*(r,z)\text{ ，}\quad(28c)$$
$$p(r,z)=\frac{Q\gamma_w}{4\pi k}\left(\frac{1}{R_a}-\frac{1}{R_b}\right)\text{ ，}\quad(28d)$$
$$\vartheta(r,z)=\frac{W}{4\pi\lambda_t}\left(\frac{1}{R_a}+\frac{1}{R_b}\right)\text{ 。}\quad(28e)$$

若考慮半平面邊界無應力變化、完全不透水、且保持隔熱狀態，則其半無限域基本解為：

$$u_r(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_1(r,z)+F_zg_2(r,z)$$
$$+Qg_3^*(r,z)+Wg_4^*(r,z)\text{，}\quad(29a)$$
$$u_\theta(r,\theta,z)=(F_\theta\cos\theta-F_r\sin\theta)g_5(r,z)\text{ ，}\quad(29b)$$
$$u_z(r,\theta,z)=(F_r\cos\theta+F_\theta\sin\theta)g_6(r,z)+F_zg_7(r,z)$$
$$+Qg_8^*(r,z)+Wg_9^*(r,z)\text{ ，}\quad(29c)$$
$$p(r,z)=\frac{Q\gamma_w}{4\pi k}\left(\frac{1}{R_a}+\frac{1}{R_b}\right)\text{ ，}\quad(29d)$$
$$\vartheta(r,z)=\frac{W}{4\pi\lambda_t}\left(\frac{1}{R_a}+\frac{1}{R_b}\right)\text{ 。}\quad(29e)$$

以上各式中之函數 $g_i(r,z)$ $(i=1,\cdots,9)$ 、 $g_i^*(r,z)$ $(i=3,4,8,9)$ 、 $R_b(r,z)$ 以及 $R_b^*(r,z)$ 等分別定義為：

$$g_1(r,z)=\frac{1}{16\pi\eta G}\left\{(2\eta+1)\frac{1}{R_a}+(2\eta-1)\frac{r^2}{R_a^3}\right.$$
$$+\frac{4\eta^2-2\eta+1}{2\eta-1}\frac{1}{R_b}-\frac{2\eta}{2\eta-1}\frac{r^2}{R_bR_b^{*2}}$$
$$\left.-2(2\eta-1)\frac{hz}{R_b^3}+(2\eta+1)\frac{r^2}{R_b^3}\right\}\text{ ，}\quad(30a)$$
$$g_2(r,z)=\frac{1}{16\pi\eta G}\left[(2\eta-1)\frac{r(z-h)}{R_a^3}-\frac{4\eta}{2\eta-1}\frac{r}{R_bR_b^*}\right.$$
$$\left.+(2\eta+1)\frac{r(z-h)}{R_b^3}+6(2\eta-1)\frac{hrz(z+h)}{R_b^5}\right]\text{，}\quad(30b)$$
$$g_3(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\left[\frac{r}{R_a}-\frac{r}{R_b^*}\right.$$
$$\left.-\frac{2hrz}{R_b^3}+\frac{2\eta+1}{2\eta-1}\frac{hr}{R_bR_b^*}-\frac{rz}{R_bR_b^*}\right]\text{，}\quad(30c)$$
$$g_3^*(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\left[\frac{r}{R_a}-\frac{r}{R_b^*}-\frac{2hrz}{R_b^3}\right.$$

$$+\frac{2\eta+1}{2\eta-1}\frac{hr}{R_b R_b^*}-(-4\eta+1)\frac{rz}{R_b R_b^*}\Bigg]\text{，} \tag{30d}$$

$$g_4(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\Bigg[\frac{r}{R_a}-\frac{r}{R_b^*}-\frac{2hrz}{R_b^3}$$

$$+\frac{2\eta+1}{2\eta-1}\frac{hr}{R_b R_b^*}-\frac{rz}{R_b R_b^*}\Bigg]\text{，} \tag{30e}$$

$$g_4^*(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\Bigg[\frac{r}{R_a}+\frac{2\eta+1}{2\eta-1}\frac{r}{R_b}-\frac{2hrz}{R_b^3}\Bigg]\text{，} \tag{30f}$$

$$g_5(r,z)=\frac{1}{16\pi\eta G}\Bigg[(2\eta+1)\frac{1}{R_a}+\frac{4\eta^2-2\eta+1}{2\eta-1}\frac{1}{R_b}$$

$$+2(2\eta-1)\frac{hz}{R_b^3}+\frac{2\eta}{2\eta-1}\frac{r^2}{R_b R_b^{*2}}\Bigg]\text{，} \tag{30g}$$

$$g_6(r,z)=\frac{1}{16\pi\eta G}\Bigg[(2\eta-1)\frac{r(z-h)}{R_a^3}-(2\eta+1)\frac{r(z-h)}{R_b^3}$$

$$+\frac{4\eta}{2\eta-1}\frac{r}{R_b R_b^*}-6(2\eta-1)\frac{hrz(z+h)}{R_b^5}\Bigg]\text{，} \tag{30h}$$

$$g_7(r,z)=\frac{1}{16\pi\eta G}\Bigg[(2\eta+1)\frac{1}{R_a}+(2\eta-1)\frac{(z-h)^2}{R_a^3}$$

$$+\frac{4\eta^2+1}{2\eta-1}\frac{1}{R_b}+(2\eta+1)\frac{(z+h)^2}{R_b^3}$$

$$-2(2\eta-1)\frac{hz}{R_b^3}+6(2\eta-1)\frac{hz(z+h)^2}{R_b^5}\Bigg]\text{，} \tag{30i}$$

$$g_8(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\Bigg[\frac{z-h}{R_a}-\frac{z}{R_b}$$

$$-\frac{2\eta+1}{2\eta-1}\frac{h}{R_b}-\frac{2hz(z+h)}{R_b^3}\Bigg]\text{，} \tag{30j}$$

$$g_8^*(r,z)=\frac{3\gamma_w(\nu_u-\nu)}{16\pi\eta kGB(1-2\nu)(1+\nu_u)}\Bigg[\frac{z-h}{R_a}-(-4\eta+1)\frac{z}{R_b}$$

$$-\frac{2\eta+1}{2\eta-1}\frac{h}{R_b}-\frac{2hz(z+h)}{R_b^3}-4\eta\sinh^{-1}\frac{z+h}{r}\Bigg]\text{，} \tag{30k}$$

$$g_9(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\Bigg[\frac{z-h}{R_a}-\frac{z}{R_b}$$

$$-\frac{2\eta+1}{2\eta-1}\frac{h}{R_b}-\frac{2hz(z+h)}{R_b^3}\Bigg]\text{，} \tag{30l}$$

$$g_9^*(r,z)=\frac{\alpha_s(1+\nu)}{8\pi\eta\lambda_t(1-2\nu)}\Bigg[\frac{z-h}{R_a}+\frac{2\eta+1}{2\eta-1}\frac{z-h}{R_b}-\frac{2hz(z+h)}{R_b^3}$$

$$-\frac{8\eta}{(2\eta-1)^2}\sinh^{-1}\frac{z+h}{r}\Bigg]\text{，} \tag{30m}$$

$$R_b=\sqrt{r^2+(z+h)^2}\text{，} \tag{30n}$$

$$R_b^*=\sqrt{r^2+(z+h)^2}+z+h\text{。} \tag{30o}$$

利用文獻[16，17]所研討出之結果並作適當的化簡，以及滲流問題與熱傳問題所具有之類比性質，可知所推導出之半無限域穩態基本解應為正確結果。

以上所研討出之無限域及半無限域穩態基本解，係以圓柱座標系統(r,θ,z)表示，並考慮作用源是作用於座標$(0,0,h)$位置所得出之結果。若引用其他座標系統或作用源係作用於其他座標位置，則基本解之函數型態僅需作適當之座標平移或旋轉等轉換，即可獲得所需之結果。

結　論

本文已研討出均質均向性之線彈性飽和多孔介質熱彈性力學理論的無限域及半無限域穩態基本解，其中之半無限域基本解的邊界條件有考慮四種不同情況，即考慮半無限域地表邊界分別為透水或不透水、恆溫或隔熱等狀態。應用此基本解，再配合已推導出之邊界積分方程式[14]，即可撰寫邊界元素計算程式，解析多孔介質之邊界值問題。

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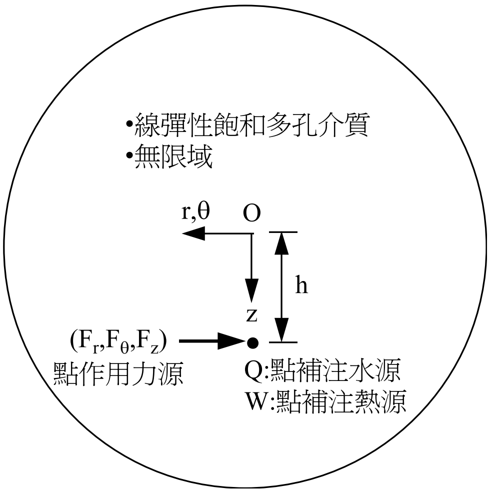

圖1　多孔介質之無限域模式圖。

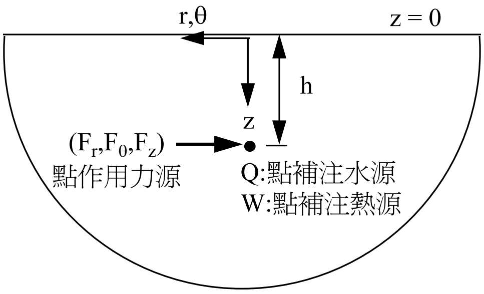

圖2　多孔介質之半無限域模式圖。

Full-Space and Half-Space Steady State Fundamental Solutions of Thermoporoelasticity

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ABSTRACT

Based on thermoporoelasticity, the isotropic linear elastic saturated porous media of full-space and half-space domains are studied. The steady state fundamental solutions due to point force, point heat source, and point water source are obtained using integral transforms. The derived three-dimensional fundamental solutions can be used as a basis of the boundary element techniques to solve the nonisothermal poroelastic problems.

Keywords: Porous Medium, Full-Space, Half-Space, Fundamental Solution.

CONSOLIDATION SETTLEMENT DUE TO A POINT SINK WITH COMPRESSIBLE CONSTITUENTS

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ABSTRACT

Based on Biot's linearized quasi-static elasticity theory of fluid-infiltrated porous materials, the constituent compressibilities of fluid and solid are taken into fully account on the mathematical modelling. The study is focused on the analytic solutions of transient consolidation due to a point sink. Using Laplace-Hankel integral transforms to solve the presented model, closed-form solutions of horizontal displacement, settlement and excess pore fluid pressure of the strata are derived. The compressibility of solid skeleton and pore fluid are important on transient consolidation deformation process. However, the long-term consolidation behaviors due to groundwater withdrawal are not directly dependent on the compressibility of poroelastic constituents of the saturated aquifer.

KEY WORDS

Closed-form Solution, Consolidation Settlement, Golden Ratio, Groundwater Withdrawal.

1. Introduction

Large amounts of groundwater withdrawal from certain types of rocks, such as fine-grained sediments, can induce land subsidence [1]. The stratum compact on itself when the groundwater is withdrawn from the saturated aquifer of the strata. As water pumps from an aquifer, the pore water pressure is reduced in the withdrawal region. It leads to increase in effective stress between the solid skeleton and thus the subsidence of ground surface.

The three-dimensional consolidation theory introduced by Biot [2,3] is generally regarded as the fundamental theory for modelling land subsidence. The approach is followed by Rice and Cleary [4] who provided an elegant formulation of Biot's theory in terms of easily identifiable quantities and material constants. Bear and Corapcioglu [5,6] presented the modified Biot's equations where the pore fluid is treated as compressible, and the solid skeleton is assumed as incompressible. Based on Biot's theory modified by Bear and Corapcioglu [5,6], Booker and Carter [7-10], Tarn and Lu [11] presented solutions of subsidence by a point sink embedded in the saturated elastic half space at a constant rate. Chen [12,13], Kanok-Nukulchai and Chau [14] presented analytic solutions for the steady-state responses of displacements and stresses in a half space subject to a fluid point sink. Lu and Lin [15,16] displayed transient displacements of the pervious half space due to steady pumping rate [15] and impulsive pumping [16]. The results presented by Hou *et al.* [17] shown that ground horizontal displacement occurred as groundwater withdrawal from an aquifer.

In the past, attempts made to model the subsidence due to groundwater withdrawal have usually employed Biot's model with compressible pore fluid. In this paper, both of solid skeleton and pore fluid are treated as compressible constituents on the mathematical model. The saturated strata are modeled as a homogeneous isotropic pervious elastic half space. The transient ground surface displacements and excess pore fluid pressure of the saturated porous aquifer due to a fluid point sink are obtained by using Laplace-Hankel integral transforms. Analytical results are illustrated and discussed to show the time dependent consolidation settlement subjected to groundwater withdrawal.

2. Mathematical Models

2.1 Basic Equations

The formulation of Biot's equations follows the pattern from Rice and Cleary [4] who provided an easily identifiable quantities and material constants. The four basic material constants selected in the constitutive equations are the shear modulus G, the drained Poisson's ratio ν, the undrained Poisson's ratio ν_u and Skempton's pore pressure coefficient B [18]. The physical ranges of B and ν_u are obviously $0 \le B \le 1$ [4] and $0 \le \nu \le \nu_u \le \frac{1}{2}$ [4], respectively. For incompressible constituents, the poroelastic coefficients $B=1$ and $\nu_u = \frac{1}{2}$. According to Rice and Cleary [4], the reformulated constitutive relations are expressed as [19]:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \frac{2G\nu}{1-2\nu}\varepsilon\delta_{ij} - \frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}p\delta_{ij}, \qquad (1)$$

$$p = -\frac{2GB(1+\nu_u)}{3(1-2\nu_u)}\varepsilon + \frac{2GB^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-2\nu_u)}\varsigma, \qquad (2)$$

in which σ_{ij}, p and ε_{ij} are the total stress components, excess pore fluid pressure and solid strain components of the poroelastic media, respectively. The fluid pressure p is positive for compression. The parameter ς is the variation of fluid content per unit reference volume. The volumetric strain of the skeletal material is denoted by ε and $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$; δ_{ij} is the Kronecker delta. The inversions of equations (1) and (2) are shown as the form:

$$\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right) + \frac{3(\nu_u-\nu)}{2GB(1+\nu)(1+\nu_u)}p\delta_{ij}, \qquad (1^*)$$

$$\varsigma = \frac{9(\nu_u-\nu)(1-2\nu_u)}{2GB^2(1-2\nu)(1+\nu_u)^2}p + \frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}\varepsilon. \qquad (2^*)$$

The solid strain components ε_{ij} and displacement components u_i are governed by the linear kinematic equation:

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right). \qquad (3)$$

The total stress components σ_{ij} must satisfy the equilibrium equations:

$$\sigma_{ij,j} + b_i = 0, \qquad (4)$$

where b_i denote the body force components. The mass balance for the fluid phase is denoted by:

$$\frac{\partial\varsigma}{\partial t} + v_{i,i} + q = 0, \qquad (5)$$

in which v_i is the specific discharge velocity components; and q is the rate of fluid extracted from the saturated porous aquifer per unit volume. Assuming that the pore fluid flow is governed by Darcy's law, we have

$$v_i = -\frac{k}{\gamma_f}p_{,i}, \qquad (6)$$

in which k denotes the permeability of the porous aquifer and γ_f is the unit weight of pore fluid.

The governing equations (1) to (6) can be combined to yield various field equations for the solutions of boundary value problems. Substituting (1) and (3) into (4), (2*) and (6) into (5), respectively, then the equilibrium equation (4) and mass balance equation (5) can be expressed in terms of displacement components u_i and excess pore fluid pressure p as below:

$$G\nabla^2 u_i + \frac{G\nu}{1-2\nu}\frac{\partial \varepsilon}{\partial x_i} - \alpha\frac{\partial p}{\partial x_i} + b_i = 0 \ , \tag{7a}$$

$$-\frac{k}{\gamma_f}\nabla^2 p + \frac{9(\nu_u-\nu)(1-2\nu_u)}{2GB^2(1-2\nu)(1+\nu_u)^2}\frac{\partial p}{\partial t} + \alpha\frac{\partial \varepsilon}{\partial t} + q = 0, \tag{7b}$$

where α is known as Biot's coefficient of effective stress which can be defined as

$$\alpha = \frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)} \ . \tag{8}$$

The above mathematical model is known as coupled model of poroelasticity where the flow field is dependent on the displacement field. The coupling term $\partial\varepsilon/\partial t$ in equation (7b) is neglected in this paper.

Figure 1 presents a fluid point sink buried in a saturated porous half space at a depth h. The constant pumping strength is denoted as Q at the location $(0,h)$. Introducing the equilibrium equations for axisymmetric poroelasticity problem with a vertical axis of symmetry and neglect the effects of body forces b_i, then equation (7a) is transformed to equations (9a) and (9b). Moreover, assuming the flow field is independent from the displacement field, then the mass balance equation (7b) can be expressed as (9c). After doing so, the uncoupled governing equations in axially symmetric coordinates (r,z) are derived in terms of displacements $u_i\,(i=r,z)$ and excess pore fluid pressure p as following:

$$G\nabla^2 u_r + \frac{G}{1-2\nu}\frac{\partial \varepsilon}{\partial r} - G\frac{u_r}{r^2} - \alpha\frac{\partial p}{\partial r} = 0 \ , \tag{9a}$$

$$G\nabla^2 u_z + \frac{G}{1-2\nu}\frac{\partial \varepsilon}{\partial z} - \alpha\frac{\partial p}{\partial z} = 0 \ , \tag{9b}$$

$$-\frac{k}{\gamma_f}\nabla^2 p + \frac{9(\nu_u-\nu)(1-2\nu_u)}{2GB^2(1-2\nu)(1+\nu_u)^2}\frac{\partial p}{\partial t} + \frac{Q}{2\pi r}\delta(r)\delta(z-h)u(t) = 0 \ , \tag{9c}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator and $\varepsilon = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$ is the volumetric strain of the poroelastic aquifer; $\delta(x)$ and $u(t)$ are the Dirac delta function and Heaviside unit step function, respectively. Equations (9a) to (9c) are the uncoupled basic field equations with a point sink at a constant pumping rate in which the fluid and solid are treated as compressible constituents.

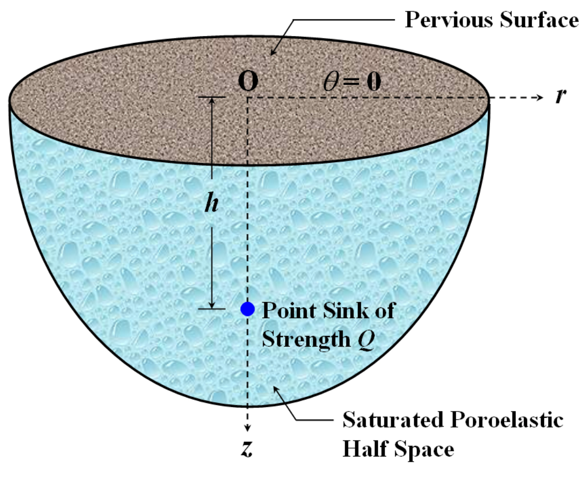

Figure 1. Poromechanics of point sink problem.

2.2 Boundary Conditions and Initial Conditions

The half space ground surface is treated as a pervious traction-free boundary for all times $t \geq 0$. Therefore, the mathematical statements of the ground surface boundary $z=0$ in axisymmetric coordinates (r,z) are:

$$\sigma_{rz}(r,0,t)=0,\ \ \sigma_{zz}(r,0,t)=0\ \text{, and}\ \ p(r,0,t)=0\ .\text{(10a)}$$

The displacements and excess pore fluid pressure at the remote boundary $z \to \infty$ due to the effect of a point sink must be nil at any time. These conditions are written as

$$\lim_{z\to\infty}\{u_r(r,z,t), u_z(r,z,t), p(r,z,t)\}=\{0,0,0\}\ . \quad (10b)$$

Assuming no initial displacements and seepage of the strata, then the initial conditions at time $t=0^+$ of the mathematical model due to a fluid point sink is treated as

$$u_r(r,z,0^+)=0,\ \ u_z(r,z,0^+)=0,\ \text{and}\ \ p(r,z,0^+)=0. \ (11)$$

The mathematical model in this study is based on the governing equations (9a) to (9c), the corresponding boundary conditions (10a)-(10b) and initial conditions (11).

3. Laplace-Hankel Transformation Analysis

Applying initial conditions in equation (11), the governing partial differential equations (9a) to (9c) are reduced to ordinary differential equations by performing Laplace-Hankel transforms [20] with respect to the time variable t and the radial coordinate r, respectively:

$$\left(\frac{d^2}{dz^2}-2\eta\xi^2\right)\tilde{u}_r-(2\eta-1)\xi\frac{d\tilde{u}_z}{dz}+\frac{\alpha}{G}\xi\tilde{p}=0\ , \quad (12a)$$

$$(2\eta-1)\xi\frac{d\tilde{u}_r}{dz}+\left(2\eta\frac{d^2}{dz^2}-\xi^2\right)\tilde{u}_z-\frac{\alpha}{G}\frac{d\tilde{p}}{dz}=0\ , \quad (12b)$$

$$-\frac{k}{\gamma_f}\left(\frac{d^2}{dz^2}-\xi^2\right)\tilde{p}+\frac{9(\nu_u-\nu)(1-2\nu_u)}{2GB^2(1-2\nu)(1+\nu_u)^2}s\tilde{p}$$
$$+\frac{Q}{2\pi s}\delta(z-h)=0, \quad (12c)$$

where ξ and s are Hankel and Laplace transform parameters. The symbols $\eta=(1-\nu)/(1-2\nu)$ and $\tilde{u}_r$, $\tilde{u}_z$, $\tilde{p}$ are defined as

$$\tilde{u}_r(z;\xi,s)=\int_0^\infty\int_0^\infty ru_r(r,z,t)exp(-st)J_1(\xi r)dtdr, \ (13a)$$

$$\tilde{u}_z(z;\xi,s)=\int_0^\infty\int_0^\infty ru_z(r,z,t)\ exp(-st)J_0(\xi r)dtdr, \ (13b)$$

$$\tilde{p}(z;\xi,s)=\int_0^\infty\int_0^\infty rp(r,z,t)\ exp(-st)J_0(\xi r)dtdr, \ (13c)$$

in which $J_a(x)$ represents the first kind of Bessel function of order a. The Laplace-Hankel inversions of equations (13a) to (13c) are:

$$u_r(r,z,t)=\frac{1}{2\pi i}\int_0^\infty\int_{\alpha-i\infty}^{\alpha+i\infty}\xi\tilde{u}_r(z;\xi,s)e^{st}J_1(\xi r)dsd\xi, \ (14a)$$

$$u_z(r,z,t)=\frac{1}{2\pi i}\int_0^\infty\int_{\alpha-i\infty}^{\alpha+i\infty}\xi\tilde{u}_z(z;\xi,s)e^{st}J_0(\xi r)dsd\xi, \ (14b)$$

$$p(r,z,t)=\frac{1}{2\pi i}\int_0^\infty\int_{\alpha-i\infty}^{\alpha+i\infty}\xi\,\tilde{p}(z;\xi,s)e^{st}J_0(\xi r)dsd\xi. \ (14c)$$

The general solutions of equations (12a) to (12c) are obtained as

$$\tilde{u}_r(z;\xi,s)=C_1\,exp(\xi z)+C_2 z\,exp(\xi z)$$
$$+C_3\,exp(-\xi z)+C_4 z\,exp(-\xi z)$$
$$+C_5\,exp\left(\sqrt{\xi^2+\frac{s}{c}}z\right)+C_6\,exp\left(-\sqrt{\xi^2+\frac{s}{c}}z\right)$$

$$+\frac{Q\alpha\gamma_f}{8\pi\eta Gk}\left[-\frac{c}{s^2}exp\left(-\xi\left|z-h\right|\right)\right.$$

$$\left.+\frac{c}{s^2}\xi\sqrt{\xi^2+\frac{s}{c}}^{-1}exp\left(-\sqrt{\xi^2+\frac{s}{c}}\left|z-h\right|\right)\right], \quad (15a)$$

$$\tilde{u}_z\left(z;\xi,s\right)=\left(-C_1+\frac{2\eta+1}{2\eta-1}\frac{1}{\xi}C_2\right)exp\left(\xi z\right)-C_2 z\,exp\left(\xi z\right)$$

$$+\left(C_3+\frac{2\eta+1}{2\eta-1}\frac{1}{\xi}C_4\right)exp\left(-\xi z\right)+C_4 z\,exp\left(-\xi z\right)$$

$$-\frac{1}{\xi}\sqrt{\xi^2+\frac{s}{c}}C_5\,exp\left(\sqrt{\xi^2+\frac{s}{c}}z\right)$$

$$+\frac{1}{\xi}\sqrt{\xi^2+\frac{s}{c}}C_6\,exp\left(-\sqrt{\xi^2+\frac{s}{c}}z\right)$$

$$\mp\frac{Q\alpha\gamma_f}{8\pi\eta Gk}\left[\frac{c}{s^2}exp\left(-\xi\left|z-h\right|\right)\right.$$

$$\left.-\frac{c}{s^2}exp\left(-\sqrt{\xi^2+\frac{s}{c}}\left|z-h\right|\right)\right], \quad (15b)$$

$$\tilde{p}\left(z;\xi,s\right)=-2\eta G\frac{1}{\xi}\frac{s}{c}C_5\,exp\left(\sqrt{\xi^2+\frac{s}{c}}z\right)$$

$$-2\eta G\frac{1}{\xi}\frac{s}{c}C_6\,exp\left(-\sqrt{\xi^2+\frac{s}{c}}z\right)$$

$$-\frac{Q\alpha\gamma_f}{4\pi k}\frac{1}{s}\sqrt{\xi^2+\frac{s}{c}}^{-1}exp\left(-\sqrt{\xi^2+\frac{s}{c}}\left|z-h\right|\right), \quad (15c)$$

where the parameter $c=\frac{2GB^2\left(1-2\nu\right)\left(1+\nu_u\right)^2}{9\left(\nu_u-\nu\right)\left(1-2\nu_u\right)}\frac{k}{\gamma_f}$ and the constants $C_i\left(i=1,2,\cdots,6\right)$ are functions of the transformed variables ξ and s. These variables are determined from the transformed boundary conditions. The upper and lower signs in equation (15b) are for the conditions of $\left(z-h\right)\geq 0$ and $\left(z-h\right)<0$, respectively.

The constitutive relations (1) and linear kinematic equation (3) for axially symmetric deformation, i.e., $\varepsilon_{rr}=\frac{\partial u_r}{\partial r}$, $\varepsilon_{\theta\theta}=\frac{u_r}{r}$ and $\varepsilon_{zz}=\frac{\partial u_z}{\partial z}$, are used to reformulate the half space boundary conditions in equation (10a). After doing so, the Laplace-Hankel transforms are applied to (10a) and (10b) with respect to the time variable t and radial coordinate r, respectively. The mechanical and hydraulic boundary conditions at $z=0$ and $z\to\infty$ of the transformed domains $\left(z;\xi,s\right)$ are derived as follows:

$$\eta\frac{d\tilde{u}_z\left(0;\xi,s\right)}{dz}+\left(\eta-1\right)\xi\tilde{u}_r\left(0;\xi,s\right)=0\,, \quad (16a)$$

$$\frac{d\tilde{u}_r\left(0;\xi,s\right)}{dz}-\xi\tilde{u}_z\left(0;\xi,s\right)=0\,, \quad (16b)$$

$$\tilde{p}\left(0;\xi,s\right)=0\,, \quad (16c)$$

$$\lim_{z\to\infty}\left\{\tilde{u}_r\left(z;\xi,s\right),\tilde{u}_z\left(z;\xi,s\right),\tilde{p}\left(z;\xi,s\right)\right\}=\left\{0,0,0\right\}, \quad (16d)$$

where $\tilde{u}_r$, $\tilde{u}_z$ and $\tilde{p}$ follow the definitions shown in equations (13a) to (13c).

The constants $C_i\left(i=1,2,\cdots,6\right)$ of the general solutions are determined by the transformed half space boundary conditions at $z=0$ and $z\to\infty$ as shown in equations (16a) to (16d). Finally, the desired quantities u_r, u_z and p are obtained by applying appropriate inverse Laplace-Hankel transformations [21,22].

The focus of the study is on the ground surface horizontal displacement $u_r\left(r,0,t\right)$, settlement $u_z\left(r,0,t\right)$ and excess pore fluid pressure $p\left(r,z,t\right)$ of the strata due to a point sink. The transformed ground surface displacements and excess pore fluid pressure of the strata are derived from equations (15a) to (15c) with the transformed boundary conditions (16a) to (16d), and they are obtained as follows:

$$\tilde{u}_r\left(0;\xi,s\right)=\frac{Q\gamma_f\left(1-2\nu\right)}{2\pi Gk}\left[-\frac{c}{s^2}exp\left(-\xi h\right)\right.$$

$$\left.+\frac{c}{s^2}exp\left(-\sqrt{\xi^2+\frac{s}{c}}h\right)\right], \quad (17a)$$

$$\tilde{u}_z\left(0;\xi,s\right)=\frac{Q\gamma_f\left(1-2\nu\right)}{2\pi Gk}\left[\frac{c}{s^2}exp\left(-\xi h\right)\right.$$

$$\left.-\frac{c}{s^2}exp\left(-\sqrt{\xi^2+\frac{s}{c}}h\right)\right], \quad (17b)$$

$$\tilde{p}(z;\xi,s) = -\frac{Q\gamma_f}{4\pi k}\left[\frac{1}{s}\sqrt{\xi^2+\frac{s}{c}}^{-1} exp\left(-\sqrt{\xi^2+\frac{s}{c}}|z-h|\right)\right.$$

$$\left.-\frac{1}{s}\sqrt{\xi^2+\frac{s}{c}}^{-1} exp\left(-\sqrt{\xi^2+\frac{s}{c}}(z+h)\right)\right]. \quad (17c)$$

Applying the Laplace-Hankel inversion formulae (14a) to (14c), equations (17a) to (17c) lead to the following transient ground surface displacements by letting $z = 0$ and the excess pore fluid pressure of strata as below:

$$u_r(r,0,t) = \frac{Q\gamma_f(1-2\nu)}{2\pi Gk}\left\{-\frac{ctr}{\left(r^2+h^2\right)^{3/2}}\right.$$

$$+\int_0^{ct}\frac{(ct-\tau)hr}{16\tau^3} exp\left(-\frac{r^2+2h^2}{8\tau}\right)$$

$$\left.\times\left[I_0\left(\frac{r^2}{8\tau}\right)-I_1\left(\frac{r^2}{8\tau}\right)\right]d\tau\right\}, \quad (18a)$$

$$u_z(r,0,t) = \frac{Q\gamma_f(1-2\nu)}{2\pi Gk}\left\{\frac{cth}{\left(r^2+h^2\right)^{3/2}} erf\left(\frac{\sqrt{r^2+h^2}}{2\sqrt{ct}}\right)\right.$$

$$-\frac{h}{r^2+h^2}\sqrt{\frac{ct}{\pi}} exp\left(-\frac{r^2+h^2}{4ct}\right)$$

$$\left.+\frac{h}{2\sqrt{r^2+h^2}} erfc\left(\frac{\sqrt{r^2+h^2}}{2\sqrt{ct}}\right)\right\}, \quad (18b)$$

$$p(r,z,t) = -\frac{Q\gamma_f}{4\pi k}\left\{\frac{1}{\sqrt{r^2+(z-h)^2}} erfc\left(\frac{\sqrt{r^2+(z-h)^2}}{2\sqrt{ct}}\right)\right.$$

$$\left.-\frac{1}{\sqrt{r^2+(z+h)^2}} erfc\left(\frac{\sqrt{r^2+(z+h)^2}}{2\sqrt{ct}}\right)\right\}. \quad (18c)$$

Here, $erf(x)$ and $erfc(x)$ are known as error function and complementary error function, respectively. The complementary error function $erfc(x)$ is defined as $erfc(x) = 1 - erf(x)$. The symbol $I_\nu(x)$ represents the modified Bessel function of the first kind of order ν.

The long-term ground surface horizontal displacement $u_r^\infty(r,0)$, settlement $u_z^\infty(r,0)$ and excess pore fluid pressure $p^\infty(r,z)$ of the aquifer are obtained when $t \to \infty$. It leads to:

$$u_r^\infty(r,0) = -\frac{Q\gamma_f(1-2\nu)}{4\pi Gk}\frac{rh}{\sqrt{r^2+h^2}\left(\sqrt{r^2+h^2}+h\right)}, \quad (19a)$$

$$u_z^\infty(r,0) = \frac{Q\gamma_f(1-2\nu)}{4\pi Gk}\frac{h}{\sqrt{r^2+h^2}}, \quad (19b)$$

$$p^\infty(r,z) = -\frac{Q\gamma_f}{4\pi k}\left[\frac{1}{\sqrt{r^2+(z-h)^2}}-\frac{1}{\sqrt{r^2+(z+h)^2}}\right]. \quad (19c)$$

The equations (19a) to (19c) show that the long-term consolidation behaviors are not directly dependent on the poroelastic coefficients B and ν_u. However, the poroelastic constants B and ν_u can affect the transient consolidation deformation process as shown in equations (18a) to (18c) through the parameter of constant c.

The maximum long-term ground surface horizontal displacement $u_{r\max}^\infty$ and settlement $u_{z\max}^\infty$ of the half space due to a point sink are derived from equations (19a) and (19b) by letting $r = \sqrt{\phi}h$ and $r = 0$, respectively. These displacements are

$$u_{r\max}^\infty = u_r\left(\sqrt{\phi}h, 0, \infty\right) = -\frac{Q\gamma_f(1-2\nu)}{4\pi Gk}\frac{1}{\phi^{2.5}}, \quad (20a)$$

$$u_{z\max}^\infty = u_z(0,0,\infty) = \frac{Q\gamma_f(1-2\nu)}{4\pi Gk}, \quad (20b)$$

in which $\phi = \left(1+\sqrt{5}\right)/2 \approx 1.618$ is known as golden ratio [23,24]. The value $r = \sqrt{\phi}h$ is derived when the slope of horizontal displacement $du_r^\infty(r,0)/dr$ becomes zero. Completing the differentiation procedure, the slope of displacement becomes

$$\frac{du_r^\infty(r,0)}{dr} = -\frac{Q\gamma_f(1-2\nu)}{4\pi Gk}\frac{hR(h^2-r^2)+h^4}{R^3(R+h)^2} = 0 \ . \quad (21)$$

Here, the distance parameter $R=\sqrt{h^2+r^2}$. This leads to the four solutions of $r=\pm\sqrt{(1+\sqrt{5})/2}h$ and $r=\pm\sqrt{(1-\sqrt{5})/2}h$. However, only $r=\sqrt{(1+\sqrt{5})/2}h$ is realistic for the radial variable $r \geq 0$.

It is interesting to find that the golden ratio ϕ appears not only in the point sink induced maximum ground surface horizontal displacement but also on the corresponding settlement by letting $r=\sqrt{\phi}h$ in equation (19b). Hence, we have:

$$u_z^\infty\left(\sqrt{\phi}h,0\right) = \frac{Q\gamma_f(1-2\nu)}{4\pi Gk}\frac{1}{\phi} \approx 0.618\, u_{z\,\max}^\infty \ . \quad (22)$$

The inverse of the golden ratio ϕ is approximately 0.618. Equation (22) shows that the maximum settlement is around 61.8% of the maximum settlement $u_{z\,\max}^\infty$, and the maximum ground surface horizontal displacement is approximately 30% of the maximum ground surface settlement from equations (20a) and (20b) as below,

$$\left|\frac{u_{r\,\max}^\infty}{u_{z\,\max}^\infty}\right| = \frac{1}{\phi^{2.5}} \approx 0.3003 \quad \text{at} \quad r=\sqrt{\phi}h \ . \quad (23)$$

The study indicates that the horizontal displacement should be properly considered in the prediction of land subsidence induced by pumping of groundwater.

4. Conclusions

The transient and long-term closed-form solutions of consolidation behavior due to a point fluid sink in the poroelastic half space were obtained by using Laplace-Hankel transformations. The results show:

1. The compressibility of solid skeleton and pore fluid are important on transient consolidation deformation process. However, the long-term consolidation behaviors are not directly dependent on the compressibility of poroelastic constituents of the saturated aquifer.
2. The maximum ground surface horizontal displacement is approximately 30% of the maximum ground surface settlement at $r=\sqrt{\phi}h \approx 1.272h$, where $\phi=(1+\sqrt{5})/2 \approx 1.618$ is known as the golden ratio. It indicates that the horizontal displacement should be properly considered in the prediction of land subsidence induced by groundwater withdrawal.

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Nomenclature

b_i	Body forces (*Pa/m*)
B	Skempton's pore pressure coefficient (Dimensionless)
c	Parameter, $c=\dfrac{2GB^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-2\nu_u)}\dfrac{k}{\gamma_f}$ (*m*²/*s*)
$erf(x)$	Error function (Dimensionless)
$erfc(x)$	Complementary error function (Dimensionless)
G	Shear modulus of the isotropic porous aquifer (N/m^2)
h	Pumping depth of the sink point (*m*)
$I_a(x)$	Modified Bessel function of the first kind of order a (Dimensionless)
$J_a(x)$	Bessel function of the first kind of order a (Dimensionless)
k	Permeability of the isotropic porous aquifer (*m*/*s*)
p	Excess pore water pressure (N/m^2)
$\tilde{p}$	Laplace-Hankel transforms of p (*Ns*)
$p^\infty(r,z)$	Long-term excess pore fluid pressure of the aquifer (N/m^2)
q	Rate of fluid extracted from the saturated porous aquifer per unit volume (s^{-1})
Q	Pumping strength of the point sink (m^3/s)
(r,θ,z)	Cylindrical coordinates system (*m*, *radian*, *m*)
s	Laplace transform parameter (s^{-1})
t	Time variable (*s*)
$u(t)$	Heaviside unit step function (Dimensionless)
u_i	Displacement components of the poroelastic aquifer (*m*)
u_r, u_z	Radial/axial displacement of the porous aquifer (*m*)
$\tilde{u}_r, \tilde{u}_z$	Laplace-Hankel transforms of u_r/u_z (m^3s)
$u_r^\infty(r,0)$	Long-term ground surface horizontal displacement (*m*)
$u_{r\max}^\infty$	Maximum ground surface horizontal displacement (*m*)
$u_z^\infty(r,0)$	Long-term ground surface settlement (*m*)
$u_{z\max}^\infty$	Maximum ground surface settlement (*m*)
v_i	Specific discharge velocity components (*m/s*)
α	Biot's coefficient of effective stress (Dimensionless)
γ_f	Unit weight of pore fluid (N/m^3)
$\delta(x)$	Dirac delta function (m^{-1})
δ_{ij}	Kronecker delta (Dimensionless)
ε	Volume strain of the porous aquifer (Dimensionless)

ε_{ij}	Strain components of the poroelastic medium (Dimensionless)
ς	Variation of fluid content per unit reference volume (Dimensionless)
η	Mechanical parameter, $\eta = (1-\nu)/(1-2\nu)$ (Dimensionless)
ν	Poisson's ratio of the isotropic porous strata (Dimensionless)
ν_u	Undrained Poisson's ratio of the poroelastic medium (Dimensionless)
ξ	Hankel transform parameter (m^{-1})
σ_{ij}	Stress components of the porous strata (N/m^2)
ϕ	Golden ratio, $\phi \approx 1.6180339887...$ (Dimensionless)