提要 140：與 *Bessel* 函數有關之進階積分式

Hankel 積分轉換與 *Bessel* 函數有關，在筆者的研究工作中發現，有 18 個重要的 *Hankel* 積分轉換公式是常會用到的。目前讀者並不會覺得這 18 個公式有什麼重要性，一般的考試也絕對不會考，但這些公式對筆者的研究工作而言是相當重要的，為方便查考起見，仍將其列為重要整理之一。表 1 為 18 個重要的 *Hankel* 積分反轉換公式。

表 1　18 個重要的 *Hankel* 積分反轉換公式

編號	積分式型態	積分結果
1	$\int_0^\infty e^{-\xi(z+h)}J_0(\xi r)\,d\xi$	$\frac{1}{R_b}$
2	$\int_0^\infty \xi e^{-\xi(z+h)}J_0(\xi r)\,d\xi$	$\frac{z+h}{R_b^3}$
3	$\int_0^\infty \xi^2 e^{-\xi(z+h)}J_0(\xi r)\,d\xi$	$-\frac{1}{R_b^3}+\frac{3(z+h)^2}{R_b^5}$
4	$\int_0^\infty e^{-\xi\lvert z-h\rvert}J_0(\xi r)\,d\xi$	$\frac{1}{R_a}$
5	$\int_0^\infty \xi e^{-\xi\lvert z-h\rvert}J_0(\xi r)\,d\xi$	$\frac{\lvert z-h\rvert}{R_a^3}$
6	$\int_0^\infty e^{-\xi(z+h)}J_1(\xi r)\,d\xi$	$\frac{r}{R_b R_b^*}$
7	$\int_0^\infty \xi e^{-\xi(z+h)}J_1(\xi r)\,d\xi$	$\frac{r}{R_b^3}$
8	$\int_0^\infty \xi e^{-\xi(z+h)}J_1(\xi r)\,d\xi$	$\frac{3r(z+h)}{R_b^5}$
9	$\int_0^\infty \frac{1}{\xi} e^{-\xi(z+h)}J_1(\xi r)\,d\xi$	$\frac{r}{R_b^*}$
10	$\int_0^\infty \xi e^{-\xi\lvert z-h\rvert}J_1(\xi r)\,d\xi$	$\frac{r}{R_a^3}$
11	$\int_0^\infty \frac{1}{\xi} e^{-\xi\lvert z-h\rvert}J_1(\xi r)\,d\xi$	$\frac{r}{R_a^*}$
12	$\int_0^\infty e^{-\xi\lvert z-h\rvert}J_1(\xi r)\,d\xi$	$\frac{r}{R_a R_a^*}$

13	$\int_0^\infty e^{-\xi(z+h)} J_2(\xi r)\, d\xi$	$\frac{r^2}{R_b R_b^{*2}}$
14	$\int_0^\infty \xi\, e^{-\xi(z+h)} J_2(\xi r)\, d\xi$	$\frac{r^2(z+h)}{R_b^3 R_b^{*2}} + \frac{2r^2}{R_b^2 R_b^{*2}}$
15	$\int_0^\infty \xi^2\, e^{-\xi(z+h)} J_2(\xi r)\, d\xi$	$\frac{3r^2}{R_b^5}$
16	$\int_0^\infty e^{-\xi\lvert z-h\rvert} J_2(\xi r)\, d\xi$	$\frac{r^2}{R_a R_a^{*2}}$
17	$\int_0^\infty \xi\, e^{-\xi\lvert z-h\rvert} J_2(\xi r)\, d\xi$	$\frac{r^2\lvert z-h\rvert}{R_a^3 R_a^{*2}} + \frac{2r^2}{R_a^2 R_a^{*2}}$
18	$\int_0^\infty \frac{1}{\xi} e^{-\mu_i \xi \lvert z\rvert} J_0(\xi r)\, d\xi$	$-\sinh^{-1}\frac{\mu_i \lvert z\rvert}{r}$

註：

1. $R_a = \sqrt{r^2 + (z-h)^2}$ ，$R_b = \sqrt{r^2 + (z+h)^2}$ ，$R_a^* = R_a + \lvert z-h\rvert$ ，$R_b^* = R_b + (z+h)$ 。

2. $\int_0^\infty x^\nu e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{(2\beta)^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}}$ ，$\mathrm{Re}\,\nu > -\frac{1}{2}$

3. $\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\sqrt{\alpha^2+\beta^2}}$ ，$\mathrm{Re}\,\alpha > \mathrm{Im}\lvert\beta\rvert$，$\mathrm{Re}\,\nu > -1$，$\mathrm{Re}(\alpha \pm i\beta) > 0$

4. $\int_0^\infty \frac{1}{x} e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\nu\beta^\nu} = \frac{\beta^\nu}{\nu\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)^\nu}$ ，$\mathrm{Re}\,\alpha > \lvert\mathrm{Im}\,\beta\rvert$ ，$\mathrm{Re}\,\nu > 0$

5. $\int_0^\infty e^{-\alpha x} J_0(\beta x)\, dx = \frac{1}{(\alpha^2+\beta^2)^{1/2}}$

6. $\int_0^\infty x e^{-\alpha x} J_0(\beta x)\, dx = \frac{\alpha}{(\alpha^2+\beta^2)^{3/2}}$ (對上式中之 α 微分一次)

7. $\int_0^\infty x^2 e^{-\alpha x} J_0(\beta x)\, dx = -\frac{1}{(\alpha^2+\beta^2)^{3/2}} + \frac{3\alpha^2}{(\alpha^2+\beta^2)^{5/2}}$ (再對上式中之 α 微分一次)

8. $\int_0^\infty x e^{-\alpha x} J_1(\beta x)\, dx = \frac{\beta}{(\alpha^2+\beta^2)^{3/2}}$

9. $\int_0^\infty x^2 e^{-\alpha x} J_1(\beta x)\, dx = \frac{3\alpha\beta}{(\alpha^2+\beta^2)^{5/2}}$ (對上式中之 α 微分一次)

10. $\int_0^\infty e^{-\alpha x} J_2(\beta x)\,dx = \dfrac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^2}{\beta^2\sqrt{\alpha^2+\beta^2}} = \dfrac{\beta^2}{\sqrt{\alpha^2+\beta^2}\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)^2}$

11. $\int_0^\infty x e^{-\alpha x} J_2(\beta x)\,dx = \dfrac{\alpha\beta^2}{\left(\alpha^2+\beta^2\right)^{3/2}\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)^2} + \dfrac{2\beta^2}{\left(\alpha^2+\beta^2\right)\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)^2}$ (對上式中之α微分一次)

12. $\Gamma(n+1) = n\Gamma(n)$ ， $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

13. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2}$

14. $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \dfrac{3}{2}\dfrac{\sqrt{\pi}}{2} = \dfrac{3\sqrt{\pi}}{4}$

15. $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \dfrac{5}{2}\dfrac{3\sqrt{\pi}}{4} = \dfrac{15\sqrt{\pi}}{8}$

16. $\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \dfrac{7}{2}\dfrac{15\sqrt{\pi}}{8} = \dfrac{105\sqrt{\pi}}{16}$

17. $\dfrac{d^n\Gamma(x)}{dx^n} = \int_0^\infty t^{x-1} e^{-t} (\ln t)^n\,dt$

18. $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x)$

19. $\Gamma(x)\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) = (2\pi)^{(m-1)/2}\, m^{1/2-mx}\,\Gamma(mx)$

20. $\Gamma(1-x)\Gamma(x) = \dfrac{\pi}{\sin(\pi x)}$

21. $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$