提要 136：貝色函數(*Bessel Function*)之各種基本關係式

貝色函數(*Bessel Function*)有許多基本而重要的關係式，條列如以下所示。

貝色函數(*Bessel Function*)之基本而重要的關係式
1. $J_{-n}(x)=(-1)^n J_n(x)$，(n = 1, 2, 3, …)
2. $J_0'(x)=-J_1(x)$
3. $J_1'(x)=J_0(x)-\frac{1}{x}J_1(x)$
4. $J_2'(x)=\frac{1}{2}[J_1(x)-J_3(x)]=\left(1-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$
5. $J_0(x)=1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+-\cdots$
6. $J_1(x)=\frac{x}{2}-\frac{x^3}{16}+\frac{x^5}{384}-\frac{x^7}{18432}+-\cdots$
7. $J_0(0)=1$
8. $J_1(0)=0$
9. $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$
10. $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$
11. $J_{3/2}(x)=\frac{1}{x}J_{1/2}(x)-J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$
12. $J_{-3/2}(x)=-\frac{1}{x}J_{-1/2}(x)-J_{1/2}(x)=-\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x}+\sin x\right)$
13. $\int x^{\nu}J_{\nu-1}(x)dx=x^{\nu}J_{\nu}(x)+C$
14. $\int x^{-\nu}J_{\nu+1}(x)dx=-x^{-\nu}J_{\nu}(x)+C$

15. $\int J_{\nu+1}(x)dx = \int J_{\nu-1}(x)dx - 2J_{\nu}(x)$

16. $J_{\nu}(x) = x^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}$

17. $Y_{\nu}(x) = \frac{2}{\pi}J_{\nu}(x)\left(\ln\frac{x}{2}+\gamma\right)+\frac{x^{\nu}}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m+h_{m+\nu})}{2^{2m+\nu}m!(m+\nu)!}x^{2m}-\frac{x^{-\nu}}{\pi}\sum_{m=0}^{\nu-1}\frac{(\nu-m-1)!}{2^{2m-\nu}m!}x^{2m}$

其中 $h_0=0$ ，$h_s=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{s}$ ；$\gamma=0.57721566490\cdots$ 。

18. $H_{\nu}^{(1)}(x) = J_{\nu}(x)+iY_{\nu}(x)$

19. $H_{\nu}^{(2)}(x) = J_{\nu}(x)-iY_{\nu}(x)$

20. $I_{\nu}(x) = \sum_{m=0}^{\infty}\frac{x^{2m+\nu}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}$

21. $Y_0(x) = \frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2}+\gamma\right)+\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m+\nu}(m!)^2}x^{2m}$

22. $Y_{\nu}(x) = \frac{1}{\sin\nu\pi}\left[J_{\nu}(x)\cos\nu\pi - J_{-\nu}(x)\right]$ ，$Y_n(x) = \lim_{\nu\to n}Y_{\nu}(x)$

23. $Y_{-n}(x) = (-1)^n Y_n(x)$

24. $I_{\nu}(x) = i^{-\nu}J_{\nu}(ix)$ ，$i=\sqrt{-1}$

25. 當 x 之值很大時，

$J_n(x)\approx\sqrt{2/(\pi x)}\cos\left(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)$ ，$Y_n(x)\approx\sqrt{2/(\pi x)}\sin\left(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\right)$

26. $\frac{d}{dx}\left[x^{\nu}J_{\nu}(x)\right] = x^{\nu}J_{\nu-1}(x)$

27. $\frac{d}{dx}\left[x^{-\nu}J_{\nu}(x)\right] = -x^{-\nu}J_{\nu+1}(x)$

28. $J_{\nu-1}(x)-J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x)$

29. $J_{\nu-1}(x)-J_{\nu+1}(x)=2J_{\nu}'(x)$

30. $\Gamma(\nu+1)=\nu\Gamma(\nu)$ ， $\nu\geq 0$

31. $\Gamma(\nu)=\int_0^{\infty}e^{-t}t^{\nu-1}dt$ ，其中 $\nu>0$

32. $\Gamma(\nu)=\int_0^{\infty}e^{-t}t^{\nu-1}dt$

33. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

34. $\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$

35. $\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\frac{\sqrt{\pi}}{2}=\frac{3\sqrt{\pi}}{4}$

36. $\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\frac{3\sqrt{\pi}}{4}=\frac{15\sqrt{\pi}}{8}$

37. $\Gamma\left(\frac{9}{2}\right)=\Gamma\left(\frac{7}{2}+1\right)=\frac{7}{2}\Gamma\left(\frac{7}{2}\right)=\frac{7}{2}\frac{15\sqrt{\pi}}{8}=\frac{105\sqrt{\pi}}{16}$

38. $\frac{d^n\Gamma(x)}{dx^n}=\int_0^{\infty}t^{x-1}e^{-t}(\ln t)^n\,dt$

39. $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\sqrt{\pi}\Gamma(2x)$

40. $\Gamma(x)\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right)=(2\pi)^{(m-1)/2}\,m^{1/2-mx}\Gamma(mx)$

41. $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin(\pi x)}$

42. $\Gamma(1)=\int_0^{\infty}e^{-t}dt=1$