提要 57：以參數變換法解析高階非齊性 ODE 之特解

幾乎所有待定係數法(*Undetermined Coefficient Method*)解不出的滿足非齊性項(*Non-homogeneous Term*)之特解(*Particular Solution*)，參數變換法(*Variation of Parameters*)都可以解得出來。茲考慮廣義之高階非齊性常微分方程式如以下所示：

$$\frac{d^n y}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_1(x)\frac{dy}{dx}+p_0(x)y=r(x) \tag{1}$$

其中 $p_0(x)$、$p_1(x)$、…、$p_{n-1}(x)$稱爲微分方程式之係數(*Coefficient*)；$r(x)$稱爲微分方程式之非齊性項(*Non-homogeneous Term*)。這種包含非齊性項之非齊性微分方程式的通解(*General Solution*) y 會包含兩部分：齊性解(*Homogeneous Solution*) y_h 跟非齊性解(*Non-homogeneous Solution*) y_p，亦即：

$$\boxed{\text{通解 } y=\text{ 齊性解 } y_h+\text{ 非齊性解 } y_p} \tag{2}$$

• 齊性解的解析

齊性解 y_h 是由齊性微分方程式 $\frac{d^n y}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_1(x)\frac{dy}{dx}+p_0(x)y=0$ 研討出，通常只需瞭解兩種類型的二階齊性微分方程式的解法即可，一種類型是係數爲常數之常係數齊性微分方程式 $\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0$，另一種類型是 *Euler-Cauchy* 微分方程式 $x^n\frac{d^n y}{dx^n}+a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1x\frac{dy}{dx}+a_0y=0$。之前已介紹過這兩種類型齊性微分方程式的解法，今再作表整理說明如下：

表一　常係數齊性微分方程式與 *Euler-Cauchy* 方程式之解析方法的比較

常係數齊性微分方程式 $\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0 y=0$	*Euler-Cauchy* 方程式 $x^n\frac{d^n y}{dx^n}+a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1 x\frac{dy}{dx}+a_0 y=0$
考慮 $y=e^{\lambda x}$	考慮 $y=x^m$
特徵方程式爲： $\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0=0$	特徵方程式爲： $m(m-1)\cdots(m-n+1)$ $+a_{n-1}m(m-1)\cdots(m-n+1)+\cdots+a_1 m+a_0=0$
特徵根爲： λ_1、λ_2、…、λ_n	特徵根爲： m_1、m_2、…、m_n
齊性解爲： 1. $\lambda_1,\cdots,\lambda_n$ 爲相異根 $y_h=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}+\cdots+C_ne^{\lambda_n x}$ 2. λ_1 與 λ_2 爲重根，其他爲相異根 $y_h=C_1e^{\lambda_1 x}+C_2xe^{\lambda_1 x}+\cdots+C_ne^{\lambda_n x}$ 3. $\lambda_1=\lambda_2=\lambda_3$，其他爲相異根 $y_h=C_1e^{\lambda_1 x}+C_2xe^{\lambda_1 x}+C_2x^2e^{\lambda_1 x}+\cdots$ $+C_ne^{\lambda_n x}$	齊性解爲： 1. $m_1,\cdots,m_n$ 爲相異根 $y_h=C_1x^{m_1}+C_2x^{m_2}+\cdots+C_nx^{m_n}$ 2. m_1 與 m_2 爲重根，其他爲相異根 $y_h=C_1x^{m_1}+C_2x^{m_1}\ln x+\cdots+C_nx^{m_n}$ 3. $m_1=m_2=m_3$，其他爲相異根 $y_h=C_1x^{m_1}+C_2x^{m_1}\ln x+C_2x^{m_1}(\ln x)^2+\cdots$ $+C_nx^{m_n}$

• 非齊性解的解析—參數變換法(*Variation of Parameters*)

非齊性解是由非齊性微分方程式 $\frac{d^n y}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_1(x)\frac{dy}{dx}+p_0(x)y=r(x)$ 研討出，如標題所示，本單元旨在介紹如何以參數變換法(*Variation of Parameters*)解析出問題的非齊性解。若 $y_1,y_2,\cdots,y_n$ 爲齊性解 y_h 中之基底(*Basis*)，則非齊性解 y_p 需與 $y_1,y_2,\cdots,y_n$ 呈線性獨立(*Linear Independence*)之關係，亦即：

$$\frac{y_p}{y_1}=u_1(x)\text{、}\frac{y_p}{y_2}=u_2(x)\text{、}\ldots\text{、}\frac{y_p}{y_n}=u_n(x) \tag{3}$$

或

$$y_p=u_1y_1\text{、}y_p=u_2y_2\text{、}\ldots\text{、}y_p=u_ny_n \tag{3'}$$

式(3')亦可改寫爲：

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \tag{4}$$

只要$u_1, u_2, \cdots, u_n$可研討出，則非齊性解y_p即可推導出。以下說明$u_1, u_2, \cdots, u_n$之解析方式。將式(4)代回式(1)，則：

$$\begin{aligned}
&\frac{d^n(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx^n} + p_{n-1}(x)\frac{d^{n-1}(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx^{n-1}} + \cdots \\
&+ p_2(x)\frac{d^2(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx^2} + p_1(x)\frac{d(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx} \\
&+ p_0(x)(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n) = r(x)
\end{aligned} \tag{5}$$

上式中之$\dfrac{d(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx}$可繼續化簡爲：

$$\begin{aligned}
\frac{d(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx} &= u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx} + \cdots + u_n\frac{dy_n}{dx} \\
&+ y_1\frac{du_1}{dx} + y_2\frac{du_2}{dx} + \cdots + y_n\frac{du_n}{dx}
\end{aligned} \tag{6}$$

若考慮

$$\boxed{y_1\frac{du_1}{dx} + y_2\frac{du_2}{dx} + \cdots + y_n\frac{du_n}{dx} = 0} \tag{7}$$

則式(6)可改寫爲：

$$\frac{d(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx} = u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx} + \cdots + u_n\frac{dy_n}{dx} \tag{6'}$$

此外，式(5)中之二次微分項$\dfrac{d^2(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx^2}$可繼續改寫爲：

$$\begin{aligned}
\frac{d^2(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n)}{dx^2} &= \frac{d}{dx}\left(u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx} + \cdots + u_n\frac{dy_n}{dx}\right) \\
&= u_1\frac{d^2 y_1}{dx^2} + u_2\frac{d^2 y_2}{dx^2} + \cdots + u_n\frac{d^2 y_n}{dx^2} \\
&+ \frac{du_1}{dx}\frac{dy_1}{dx} + \frac{du_2}{dx}\frac{dy_2}{dx} + \cdots + \frac{du_n}{dx}\frac{dy_n}{dx}
\end{aligned} \tag{8}$$

若考慮：

$$\boxed{\frac{du_1}{dx}\frac{dy_1}{dx} + \frac{du_2}{dx}\frac{dy_2}{dx} + \cdots + \frac{du_n}{dx}\frac{dy_n}{dx} = 0} \tag{9}$$

則式(8)可改寫爲：

$$\frac{d^2\left(u_1y_1+u_2y_2+\cdots+u_ny_n\right)}{dx^2}=u_1\frac{d^2y_1}{dx^2}+u_2\frac{d^2y_2}{dx^2}+\cdots+u_n\frac{d^2y_n}{dx^2} \tag{8'}$$

以上處理方式，是擬以$\frac{du_1}{dx}$、$\frac{du_2}{dx}$、…、$\frac{du_n}{dx}$等 n 個項次爲未知數，然後另找出 n 個方程式來解這 n 個未知數。目前，已經找到式(7)與式(9)這兩個方程式了。依此方式，繼續討論其餘的方程式，則倒數第二個方程式應是：

$$\boxed{\frac{du_1}{dx}\frac{d^{n-2}y_1}{dx^{n-2}}+\frac{du_2}{dx}\frac{d^{n-2}y_2}{dx^{n-2}}+\cdots+\frac{du_n}{dx}\frac{d^{n-2}y_n}{dx^{n-2}}=0} \tag{10}$$

且其所對應之微分項爲：

$$\frac{d^{n-1}\left(u_1y_1+u_2y_2+\cdots+u_ny_n\right)}{dx^{n-1}}=u_1\frac{d^{n-1}y_1}{dx^{n-1}}+u_2\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+u_n\frac{d^{n-1}y_n}{dx^{n-1}} \tag{11}$$

最後再將式(6’)、式(8’)、式(11)等代回式(5)，即可找出最後一個條件方程式：

$$\begin{aligned}
&\frac{d}{dx}\left(u_1\frac{d^{n-1}y_1}{dx^{n-1}}+u_2\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+u_n\frac{d^{n-1}y_n}{dx^{n-1}}\right)\\
&+p_{n-1}(x)\left(u_1\frac{d^{n-1}y_1}{dx^{n-1}}+u_2\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+u_n\frac{d^{n-1}y_n}{dx^{n-1}}\right)\\
&+\cdots\\
&+p_2(x)\left(u_1\frac{d^2y_1}{dx^2}+u_2\frac{d^2y_2}{dx^2}+\cdots+u_n\frac{d^2y_n}{dx^2}\right)\\
&+p_1(x)\left(u_1\frac{dy_1}{dx}+u_2\frac{dy_2}{dx}+\cdots+u_n\frac{dy_n}{dx}\right)\\
&+p_0(x)\left(u_1y_1+u_2y_2+\cdots+u_ny_n\right)=r(x)
\end{aligned} \tag{12}$$

其中

$$\begin{aligned}
\frac{d}{dx}\left(u_1\frac{d^{n-1}y_1}{dx^{n-1}}+u_2\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+u_n\frac{d^{n-1}y_n}{dx^{n-1}}\right)&=\frac{du_1}{dx}\frac{d^{n-1}y_1}{dx^{n-1}}+\frac{du_2}{dx}\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+\frac{du_n}{dx}\frac{d^{n-1}y_n}{dx^{n-1}}\\
&\quad+u_1\frac{d^ny_1}{dx^n}+u_2\frac{d^ny_2}{dx^n}+\cdots+u_n\frac{d^ny_n}{dx^n}
\end{aligned}$$

基於此，式(12)可化簡爲：

$$
\begin{aligned}
&\left(\frac{du_1}{dx}\frac{d^{n-1}y_1}{dx^{n-1}}+\frac{du_2}{dx}\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+\frac{du_n}{dx}\frac{d^{n-1}y_n}{dx^{n-1}}+u_1\frac{d^n y_1}{dx^n}+u_2\frac{d^n y_2}{dx^n}+\cdots+u_n\frac{d^n y_n}{dx^n}\right)\\
&+p_{n-1}(x)\left(u_1\frac{d^{n-1}y_1}{dx^{n-1}}+u_2\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+u_n\frac{d^{n-1}y_n}{dx^{n-1}}\right)\\
&+\cdots\\
&+p_2(x)\left(u_1\frac{d^2y_1}{dx^2}+u_2\frac{d^2y_2}{dx^2}+\cdots+u_n\frac{d^2y_n}{dx^2}\right)\\
&+p_1(x)\left(u_1\frac{dy_1}{dx}+u_2\frac{dy_2}{dx}+\cdots+u_n\frac{dy_n}{dx}\right)\\
&+p_0(x)\left(u_1y_1+u_2y_2+\cdots+u_ny_n\right)=r(x)
\end{aligned}
\tag{12'}
$$

上式可繼續整理為：

$$
\begin{aligned}
&\frac{du_1}{dx}\frac{d^{n-1}y_1}{dx^{n-1}}+\frac{du_2}{dx}\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+\frac{du_n}{dx}\frac{d^{n-1}y_n}{dx^{n-1}}\\
&+u_1\left[\frac{d^n y_1}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y_1}{dx^{n-1}}+\cdots+p_2(x)\frac{d^2y_1}{dx^2}+p_1(x)\frac{dy_1}{dx}+p_0(x)y_1\right]\\
&+u_2\left[\frac{d^n y_2}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+p_2(x)\frac{d^2y_2}{dx^2}+p_1(x)\frac{dy_2}{dx}+p_0(x)y_2\right]\\
&+\cdots\\
&+u_{n-1}\left[\frac{d^n y_{n-1}}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y_{n-1}}{dx^{n-1}}+\cdots+p_2(x)\frac{d^2y_{n-1}}{dx^2}+p_1(x)\frac{dy_{n-1}}{dx}+p_0(x)y_{n-1}\right]\\
&+u_n\left[\frac{d^n y_n}{dx^n}+p_{n-1}(x)\frac{d^{n-1}y_n}{dx^{n-1}}+\cdots+p_2(x)\frac{d^2y_n}{dx^2}+p_1(x)\frac{dy_n}{dx}+p_0(x)y_n\right]=r(x)
\end{aligned}
\tag{12''}
$$

其中

$$
\begin{cases}
\dfrac{d^n y_1}{dx^n}+p_{n-1}(x)\dfrac{d^{n-1}y_1}{dx^{n-1}}+\cdots+p_2(x)\dfrac{d^2y_1}{dx^2}+p_1(x)\dfrac{dy_1}{dx}+p_0(x)y_1=0\\
\dfrac{d^n y_2}{dx^n}+p_{n-1}(x)\dfrac{d^{n-1}y_2}{dx^{n-1}}+\cdots+p_2(x)\dfrac{d^2y_2}{dx^2}+p_1(x)\dfrac{dy_2}{dx}+p_0(x)y_2=0\\
\qquad\vdots\\
\dfrac{d^n y_{n-1}}{dx^n}+p_{n-1}(x)\dfrac{d^{n-1}y_{n-1}}{dx^{n-1}}+\cdots+p_2(x)\dfrac{d^2y_{n-1}}{dx^2}+p_1(x)\dfrac{dy_{n-1}}{dx}+p_0(x)y_{n-1}=0\\
\dfrac{d^n y_n}{dx^n}+p_{n-1}(x)\dfrac{d^{n-1}y_n}{dx^{n-1}}+\cdots+p_2(x)\dfrac{d^2y_n}{dx^2}+p_1(x)\dfrac{dy_n}{dx}+p_0(x)y_n=0
\end{cases}
\tag{13}
$$

故式(12”)可改寫為：

$$\boxed{\frac{du_1}{dx}\frac{d^{n-1}y_1}{dx^{n-1}}+\frac{du_2}{dx}\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+\frac{du_n}{dx}\frac{d^{n-1}y_n}{dx^{n-1}}=r(x)} \tag{14}$$

式(7) 、式(9)、式(10)與式(14)中恰好含有 n 個未知數 $\frac{du_1}{dx}$ 、$\frac{du_2}{dx}$ 、…、$\frac{du_n}{dx}$ ，且這 n 個方程式剛好可以解出這 n 個未知數。由式(7) 、式(9)、式(10)與式(14)知：

$$\boxed{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{bmatrix} \begin{Bmatrix} \frac{du_1}{dx} \\ \frac{du_2}{dx} \\ \vdots \\ \frac{du_n}{dx} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ r \end{Bmatrix}} \tag{15}$$

利用 *Cramer* 法則可求出這 n 個未知數：

$$\frac{du_1}{dx}=\frac{\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ r & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{r\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{rW_1}{W} \tag{16a}$$

$$\frac{du_2}{dx}=\frac{\begin{vmatrix} y_1 & 0 & \cdots & y_n \\ \frac{dy_1}{dx} & 0 & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & r & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{r\begin{vmatrix} y_1 & 0 & \cdots & y_n \\ \frac{dy_1}{dx} & 0 & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & 1 & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{rW_2}{W} \tag{16b}$$

$$\frac{du_n}{dx}=\frac{\begin{vmatrix} y_1 & y_2 & \cdots & 0 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{r\begin{vmatrix} y_1 & y_2 & \cdots & 0 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}}=\frac{rW_n}{W} \qquad (16c)$$

其中

$$W_1=\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix} \qquad (17a)$$

$$W_2=\begin{vmatrix} y_1 & 0 & \cdots & y_n \\ \frac{dy_1}{dx} & 0 & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & 1 & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix} \qquad (17b)$$

$$W_n=\begin{vmatrix} y_1 & y_2 & \cdots & 0 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & 1 \end{vmatrix} \qquad (17c)$$

$$W=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix} \qquad (17d)$$

由式(16a)-(16c)得知：

$$u_1=\int\frac{du_1}{dx}dx=\int\frac{rW_1}{W}dx \qquad (18a)$$

$$u_2 = \int \frac{du_2}{dx}\,dx = \int \frac{rW_2}{W}\,dx \tag{18b}$$

$$u_n = \int \frac{du_n}{dx}\,dx = \int \frac{rW_n}{W}\,dx \tag{18c}$$

由式(4)、式(18a)-(18c)知，問題之非齊性解爲：

$$\boxed{y_p = y_1 \int \frac{rW_1}{W}\,dx + y_2 \int \frac{rW_2}{W}\,dx + \cdots + y_n \int \frac{rW_n}{W}\,dx} \tag{19}$$

附註：式(19)之公式是基於式(1)之微分型態所推導出來的，而式(1)中之 $y^{(n)}$ 項次的係數爲 1，應用公式(19)時，一定要留意式(1)中之 $y^{(n)}$ 項次的係數是否爲 1，若不是 1，公式(19)就不能使用。此一要點，是背公式的同學常常忽略的，務須切記！

範例一
試解出常係數非齊性微分方程式之通解：$x^3y'''-3x^2y''+6xy'-6y=x^4\ln x$。 【94 成大土木所 20%，94 高應電子所 15%】

【解答】

• 齊性解的解析

由齊性微分方程式$x^3y'''-3x^2y''+6xy'-6y=0$解析出齊性解$y_h$。令齊性解$y_h$爲：

$$y_h=x^m$$

代回上式，則

$$x^3\frac{d^3(x^m)}{dx^3}-3x^2\frac{d^2(x^m)}{dx^2}+6x\frac{d(x^m)}{dx}-6(x^m)=0$$

上式可化簡爲：

$$[m(m-1)(m-2)-3m(m-1)+6m-6](x^m)=0$$

因爲不希望所研討出之齊性解爲零，亦即$x^m\neq 0$，故：

$$m(m-1)(m-2)-3m(m-1)+6m-6=0$$

上式可因式分解爲：

$$(m-1)(m-2)(m-3)=0$$

故問題之齊性解的特徵根(*Characteristic Root*) m 爲：

$$m=1、m=2、m=3$$

由重疊原理(*Superposition Principle*)知，問題之齊性解(*Homogeneous Solution*)爲：

$$\boxed{y_h=C_1x+C_2x^2+C_3x^3}$$

• 非齊性解的解析

本題必須採用參數變換方法求解問題之非齊性解y_p，由式(19)知，非齊性解y_p可表爲：

$$y_p=y_1\int\frac{rW_1}{W}dx+y_2\int\frac{rW_2}{W}dx+\cdots+y_n\int\frac{rW_n}{W}dx$$

然而，原微分方程式之最高次微分項y'''的係數並不是 1，故原式應先除以x^3，才能使用以上之公式。原式除以x^3後可得：

$$y'''-\frac{3}{x}y''+\frac{6}{x^2}y'-\frac{6}{x^3}y=x\ln x$$

故公式中之$r(x)$爲$x\ln x$，而不是$x^4\ln x$。其中他各項分述如下：

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}=12x^3+2x^3-6x^3-6x^3=2x^3$$

$$W_1=\begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix}=3x^4-2x^4=x^4$$

$$W_2=\begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix}=x^3-3x^3=-2x^3$$

$$W_3=\begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}=2x^2-x^2=x^2$$

故

$$\begin{aligned} y_p &= x\int\frac{(x^4)(x\ln x)}{2x^3}dx+x^2\int\frac{(-2x^3)(x\ln x)}{2x^3}dx+x^3\int\frac{(x^2)(x\ln x)}{2x^3}dx \\ &= x\int\frac{x^2\ln x}{2}dx-x^2\int x\ln x dx+x^3\int\frac{\ln x}{2}dx \\ &= \frac{x}{2}\left(\frac{x^3}{3}\ln x-\frac{x^3}{9}\right)-x^2\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)+\frac{x^3}{2}(x\ln x-x) \\ &= \frac{x^4}{6}\left(\ln x-\frac{11}{6}\right) \end{aligned}$$

因通解係齊性解 y_h 與非齊性解 y_p 的和，故問題之通解 y 爲：

$$\boxed{y=C_1x+C_2x^2+C_3x^3+\frac{x^4}{6}\left(\ln x-\frac{11}{6}\right)}$$

習題

1. Solve $x^3y''' - 6x^2y'' + 11xy' - 6y = 12 + x$.【93 中興材料所 15%】

2. Solve $x^3y''' + 6x^2y'' + 7xy' + y = 6 + \dfrac{2}{x}$.【93 高科機械所 15%】

3. Solve $x^3y''' + x^2y'' - 2xy' + 2y = x\ln x$.【93 高科光電所 10%】

4. Solve $(x+1)^3 y''' + (x+1)^2 y'' + 3(x+1)y' - 8y = \dfrac{x}{\sqrt{x+1}}$.【91 高科機械所 20%】

5. Solve $x^2y''' - 6y' = 2x^2$.【89 台科電子所 10%】

6. Solve $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4\ln x$.【91 北科機電所 10%】

7. Solve $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4e^x$.【90 高科電子所 15%】

8. Solve $y''' - 3y'' + 3y' - y = \dfrac{1}{x}e^x$.【88 成大造船所 20%，89 台大土木所 12%】

9. $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = \tan^{-1}x$ has three homogeneous solutions $y_1(x)$, $y_2(x)$ and $y_3(x)$. Define $W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$, $C_1(x) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \tan^{-1}x & y_2'' & y_3'' \end{vmatrix}$, $C_2(x) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & \tan^{-1}x & y_3'' \end{vmatrix}$, $C_3(x) = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & \tan^{-1}x \end{vmatrix}$. Prove that the particular solution $y_p(x) = y_1(x)\int \dfrac{C_1(x)}{W(x)}dx + y_2(x)\int \dfrac{C_2(x)}{W(x)}dx + y_3(x)\int \dfrac{C_3(x)}{W(x)}dx$.【90 台科電子所 15%】

10. Find the general solution for $x^3y''' - 2x^2y'' + 3xy' - 3y = 2x + 3x^3$, where y is a function of x.【95 交大土木所 20%】