提要 50：認識高階 ODE 之解的基底所對應的 *Wronskian*

已知高階之線齊性微分方程式(*Linear Homogeneous Differential Equation*)：

$$\frac{d^n y}{dx^n}+p_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+p_1(x)\frac{dy}{dx}+p_0(x)y=0 \tag{1}$$

的通解(*General Solution*)爲 $y=c_1y_1+c_2y_2+c_3y_3+\cdots+c_ny_n$，其中 c_1、c_2、…、c_n 爲任意常數；而 y_1、y_2、y_3、…、y_n 稱爲通解中之基底(*Basis*)。y_1、y_2、y_3、…、y_n 之 *Wronskian* 係定義爲如下所示之行列式的運算：

$$W(y_1,y_2,y_3,\cdots,y_n)=\begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{2}$$

利用 *Wronskian* 的定義，可以將非齊性微分方程式(*Non-homogeneous Differential Equation*)之非齊性解(*Non-homogeneous Solution*) y_p 的參數變換解析方法(*Variation of Parameters*)以簡易之型式表達出來，容後再加以說明。

習題

1. Let y_1 and y_2 be linear independent solutions of $x^2 y'' + 2xy' + (x-2)y = 0$. Then the Wronskian of y_1 and y_2 can be defined as $W(x) = y_1 y_2' - y_1' y_2$. Given $y_1(1) = 0$, $y_1'(1) = 1$, $y_2(1) = 2$, and $y_2'(1) = 3$, find $W(x)$.【88 台科電機所 10%】

2. If $u = u(x)$ and $v = v(x)$ are solutions of $\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$, then the Wronskian is defined as $W(x) = uv' - vu'$.
   (a) Show that $W'(x) = -p(x)W(x)$.
   (b) Solve the differential equation $W'(x) = -p(x)W(x)$.
   (c) Show that if $W(x_0) \neq 0$ for some point x_0 in $a < x < b$, then $W(x) \neq 0$ for all points in $a < x < b$.【88 北科電腦通訊所 15%】