單元 17　Conformal Mapping (保角轉換)

【例題 1】

Find the region onto which the half plane $y>0$ is mapped by the transformation $w=(1+i)z$ by using polar coordinates. Sketch this region.【91 中正電機】

【參考解答】當 $y>0,\ 0<r<\infty,\ 0<\theta<\pi,\ 0<R<\infty,\ \frac{\pi}{4}<\phi<\frac{5}{4}\pi$

【例題 2】

設 $f(z)=-\frac{1}{z}$，求 $|z|\le 1$、$y\ge 0$ 之映像點集合。【91 交大電物】

【參考解答】當 $y\ge 0$，$\frac{v}{u^2+v^2}\ge 0$，得 $v\ge 0$。

【例題 3】

Find the images of the circles $|z|=1$ and $|z|=2$ under the mapping $T(z)=\frac{z+2}{z-1}$, and what are the images of the interiors of these circles?【90 中央電機】

【參考解答】

當 $|z|\le 1,\ \frac{|w+2|}{|w-1|}\le 1,\ |w+2|\le|w-1|,\ 6u\le -3,\ u\le\frac{1}{2}$ 為 $|z|\le 1$ 之映像圖。

當 $|z|\le 2,\ \frac{|w+2|}{|w-1|}\le 2,\ |w+2|\le 2|w-1|,\ (u-2)^2+v^2\ge 4$ 為 $|z|\le 2$ 之映像圖。

【例題 4】

Find the mapping of the lines $x=2,\ y=-1$ under the transformation of $w=1/z$.【89 中興土木】

【參考解答】For $x=1,\ \left(u-\frac{1}{2}\right)^2+v^2=\frac{1}{4}.$

For $y=1,\ \left(v+\frac{1}{2}\right)^2+u^2=\frac{1}{4}.$

【例題 5】

Please graph the region in the *w*-plane of the image for the region $0 \le \mathrm{Re}(z) \le 1,\ 0 \le \mathrm{Im}(z) \le \pi$ in the *z*-plane under the mapping $w = e^z$.

【87 北科機電整合】

【參考解答】on $c_1,\ x: 0 \to 1,\ y = 0,\ u + iv = e^x,\ u: 1 \to e,\ v = 0$

on $c_2,\ x = 1,\ y: 0 \to \pi,\ u + iv = e[\cos y + i \sin y],\ u = e \to -e$

on $c_3,\ x: 1 \to 0,\ y = \pi,\ u + iv = -e^x,\ u = -1$

on $c_4,\ x = 0,\ y: \pi \to 0,\ u + iv = \cos y + i \sin y,\ u^2 + v^2 = 1$

【例題 6】

Map $|z| < 2$ onto the domain $D^*: u + v > 0$ in the *w* plane. 【86 台科電機】

【參考解答】$w = e^{-i\frac{\pi}{4}} \cdot i\dfrac{1 + w_1}{1 - w_1} = ie^{-i\frac{\pi}{4}}\dfrac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$，$w = e^{-i\frac{\pi}{4}} \cdot \dfrac{2 + z}{2 - z}$ 為 mapping function。

【例題 7】

Find a linear fractional transformation that maps $|z| \le 1$ onto $|w| \le 1$ such that $z = \dfrac{i}{2}$ is mapped to $w = 0$. 【91 清大電機】

【參考解答】令 $w = \dfrac{w_1 - A}{w_1 - \overline{A}} = \dfrac{(-2 + 4i)z + (2 + i)}{(-2 + i)z + (2 + 4i)}$ 為所求