單元 13　Fourier Transform

【例題 1】

Evaluate the integral $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$ by the residue theorem.【90 台大土木、91 成大機械】

【參考解答】$\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx=\pi i$，$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi$。

【例題 2】

(1) Find the roots of $1+z^4=0$ and the sum of the residues of $\exp(iz)/(1+z^4)$ in the upper half plane only.

(2) Evaluate the integral $\int_{0}^{\infty}\frac{\cos x}{1+x^4}dx$ by the contour integral and the Cauchy integral theorem.【91 台大土木、90 交大電信】

【參考解答】

(1) $\mathrm{Res}(e^{i\frac{\pi}{4}})+\mathrm{Res}(e^{i\frac{3}{4}\pi})=\frac{1}{4}e^{-\frac{1}{\sqrt{2}}}e^{i(\frac{1}{\sqrt{2}}-\frac{3}{4}\pi)}+\frac{1}{4}e^{-\frac{1}{\sqrt{2}}}e^{-i(\frac{1}{\sqrt{2}}-\frac{3}{4}\pi)}$

(2) $\int_{0}^{\infty}\frac{\cos x}{x^4+1}dx=\frac{\sqrt{2}\pi}{4}e^{-\frac{1}{\sqrt{2}}}(\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}})$

【例題 3】

Use residues theorem to evaluate the following integrations.

(1) $\int_{-\infty}^{\infty}\frac{x\sin(kx)}{x^2+a^2}dx$ (a and k are real, and $k>0$)

(2) $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx$ 【90 交大物理、90 交大機械】

【參考解答】(1) $I=\pi e^{-ak}=\int_{-\infty}^{\infty}\frac{x\sin(kx)}{x^2+a^2}dx$

(2) $I=\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{1}{4}\mathrm{Re}[\int_{-\infty}^{\infty}\frac{1-e^{i2x}}{x^2}dx]=\frac{\pi}{2}$

【例題 4】

Slove $\int_{-\infty}^{\infty}\frac{e^{ix}}{x^3-x^2+4x-4}dx$ by complex analysis (where $i=\sqrt{-1}$). 【90 中興機械】

【參考解答】$\int_{-\infty}^{\infty}\frac{e^{ix}}{x^3-x^2+4x-4}dx=-\pi i[\frac{1}{5}e^{i}+\frac{e^{-2}}{2i(1+2i)}]$

【例題 5】

Calculate the Fourier transform of $(\frac{2\sin t}{t})^2$. 【90 台科電子】

【參考解答】當 $w>2$，$\int_{-\infty}^{\infty}\frac{1}{t^2}[2e^{-iwt}-e^{i(w-2)t}-e^{i(w+2)t}]dt=0$；

當 $0<w<2$，$\int_{-\infty}^{\infty}\frac{1}{t^2}[2e^{-iwt}-e^{i(w-2)t}-e^{i(w+2)t}]dt=2\pi(2-w)$；

當 $-2<w<0$，$\int_{-\infty}^{\infty}\frac{1}{t^2}[2e^{-iwt}-e^{i(w-2)t}-e^{i(w+2)t}]dt=2\pi(w+2)$；

當 $w<-2$，$\int_{-\infty}^{\infty}\frac{1}{t^2}[2e^{-iwt}-e^{i(w-2)t}-e^{i(w+2)t}]dt=0$；

$$\therefore F[\frac{4\sin^2 t}{t^2}]=\begin{cases}0, w\ge 2\\ 2\pi(2-w), 0\le w\le 2\\ 2\pi(2+w), -2\le w\le 0\\ 0, w\le -2\end{cases}$$

【例題 6】

(1) Determine the Fourier transform of the function $f(t)=\frac{5e^{3it}}{t^2-4t+13}$.

(2) Find the inverse Fourier transform of the function

$F(w)=e^{-3|w+4|}\cos(2w+8)$. 【89 淡江電機】

【參考解答】(1) 當 $w\ge 3, \int_{-\infty}^{\infty}\frac{5e^{3i(w-3)t}}{t^2-4t+13}dt=\frac{5\pi}{3}e^{-(2i+3)(w-3)}$

當 $w<3, \int_{-\infty}^{\infty}\frac{5e^{3i(w-3)z}}{z^2-4z+13}dt=2\pi i\cdot\frac{5}{6i}e^{-(2i+3)(w-3)}$

(2) $F^{-1}[F(w)]=\frac{1}{2\pi}e^{-i4t}[\frac{3}{9+(t-2)^2}+\frac{3}{9+(t+2)^2}]$

【例題 7】

Show that $\int_0^{\infty} \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x}$, if $x > 0$. 【91 逢甲機械】

【參考解答】$\int_0^{\infty} \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x}$, $x > 0$，故得證。