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【例題 1】

Classify the singularities of $f(z)=\dfrac{1}{(z-1)(z-2)}$. Obtain the Laurent expansion centered on z =0 for the regions:

$(1)|z|<1$　$(2)1<|z|<2$　$(3)|z|>2$　$(4)0<|z-1|<1$

【參考解答】因為 $f(z)=\dfrac{1}{(z-1)(z-2)}$，所以 $z=1$、2 皆為一階極點。

(1) 當 $|z|<1$，$f(z)=\dfrac{1}{2}+\dfrac{3}{4}z+\dfrac{7}{8}z^2+\cdots$

(2) 當 $1<|z|<2$，$f(z)=-\dfrac{1}{2}[1+\dfrac{1}{2}z+\dfrac{1}{4}z^2+\cdots]-\dfrac{1}{z}[1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots]$

(3) 當 $|z|>2$，$f(z)=\dfrac{1}{z^2}+\dfrac{3}{z^2}+\dfrac{7}{z^4}+\cdots$

(4) 當 $0<|z-1|<1$，$f(z)=-\dfrac{1}{z-1}-1-(z-1)-(z-1)^2-\cdots$，$z=1$ 為一階極點。

【例題 2】

Find the Laurent series of $f(z)=\dfrac{1}{z(1+z^2)}$ around $z_0=0$ for

(1) $0<|z|<1$　(2) $1<|z|<\infty$.【91 暨南電機】

【參考解答】(1) $f(z)=\dfrac{1}{z}[1-z^2+z^4-z^6+\cdots]$

(2) $f(z)=\dfrac{1}{z^3}-\dfrac{1}{z^5}+\dfrac{1}{z^7}-\cdots$

【例題 3】

Expand $f(z)=\dfrac{2z-1}{(z+1)(z-2)}$ into Laurent series centered at $z=-1$, i.e., into power series in $z+1$. You should discuss the expansion in each regions of the complex plane and specify clearly the convergence region of each of your power series.【91 清大電機】

【參考解答】已知 $f(z)=\dfrac{2z-1}{(z+1)(z-2)}$，

(1)當 $0<|z+1|<3$ 時，$f(z)=\dfrac{1}{z+1}-\dfrac{1}{3}-\dfrac{1}{3^2}(z+1)-\dfrac{1}{3^3}(z+1)^2-\cdots$

(2)當 $3<|z+1|<\infty$ 時，$f(z)=\dfrac{2}{z+1}+\dfrac{3}{(z+1)^2}+\dfrac{3^2}{(z+1)^3}+\cdots$

$z=-1$ 為一階極點。

【例題 4】

Consider $f(z)=(z^2-1)^{-1}$ where $z=x+iy$ is a complex variable. The Laurent expansion of $f(z)$ with $z=1+i$ as the center can be expressed as $f(z)=\sum C_n(z-1-i)^n$. If the region of convergence of this expansion is $1<|z-1-i|<\sqrt{5}$, please find out the coefficients C_{-2} and C_2.【90 中央土木】

【參考解答】$C_{-2}=-\dfrac{1}{2}\dfrac{1}{(2+i)^3}$，$C_2=-\dfrac{i}{2}$。

【例題 5】

Let C denote the unit circle $|z|=1$ taken counterclockwise. Show that

$$\frac{1}{2\pi i}\oint_C \exp(z+\frac{1}{z})dz=\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\quad (0!=1).$$ 【86 清大工科】

【參考解答】$\dfrac{1}{2\pi i}\oint_C \exp(z+\dfrac{1}{z})dz=\sum_{n=0}^{\infty}\dfrac{1}{n!(n+1)}$，故得證。