單元 5　複數之線積分

【例題 1】

Evaluate $\int_c |z|^2 dz$, where C is the straight line segment from -2 to 1 in the complex plane.【90 台科電機】

【參考解答】令 $z = x + iy$，$|z|^2 = x^2 + y^2$，$\int_c |z|^2 dz = \frac{40}{12} + i\frac{40}{24} = \frac{5}{3}(2+i)$。

【例題 2】

$\int_{-i}^{i} |z| dz = ?$

(1) Integrating along a straight line segment.

(2) Integrating along the left half of the unit circle.【90 交大電控、光電】

【參考解答】(1) $\int_{-i}^{i} |z| dz = \frac{i}{2} + \frac{i}{2} = i$，(2) $\int_{-i}^{i} |z| dz = e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}} = 2i$。

【例題 3】

Evaluate the integral $\int_C z^m (\bar{z})^n dz$ where m and n are integers and C is the unit circle, $|z| = 1$, taken counterclockwise.【91 交大電控】

【參考解答】$\int_C z^m (\bar{z})^n dz = i\int_0^{2\pi} e^{i(m-n+1)\theta} d\theta = \begin{cases} 0, & m-n+1 \neq 0 \\ 2\pi i, & m-n+1 = 0 \end{cases}$

【例題 4】

Evaluate the following integral $\int_{-1}^{1} \frac{z+1}{z} dz$.

(1) If the path is the upper half of the circle $r = 1$.

(2) If the path is the lower half of the circle $r = 1$.

(3) Explain the solutions you have obtained in (1) and (2). 【89 成大電機】

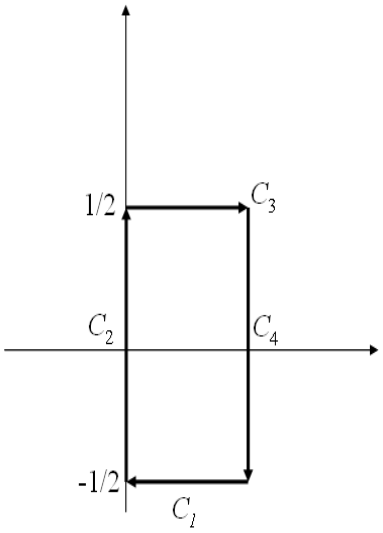
【參考解答】(1) $\int_{-1}^{1} \frac{z+1}{z^2} dz = i[\pi + \frac{2}{i}]$

(2) $\int_{-1}^{1} \frac{z+1}{z^2} dz = i[\pi - \frac{2}{i}]$

(3)解析函數有奇異點，積分值與路徑無關。

【例題 5】

Given the integreal $I = \int_C \frac{dz}{z-1}$ on the complex plane. With the contour C defined as in the following figure.



(1) Compute the integrations of the four segments of the contour C separately. Add the four results together to get I.
(2) Use the Residue Theorem to compute the integration I. Check the result with that obtains in (1).

【參考解答】(1)on C_1 $\int_{C_1} \frac{dz}{z-1} = \frac{1}{2}\ln 5 - i\tan^{-1} 2$

on C_2 $\int_{C_2} \frac{dz}{z-1} = -i \cdot 2\tan^{-1}\frac{1}{2}$

on C_3 $\int_{C_3} \frac{dz}{z-1} = -\frac{1}{2}\ln 5 - i\tan^{-1} 2$

on C_4 $\int_{C_4} \frac{dz}{z-1} = -\pi i$

(2)定義積分主值，可得 $\int_C \frac{dz}{z-1} = -\pi i$ 。