單元2　多值函數、分枝、分枝點與分枝切割

【例題1】

(1) Prove that if an analytic function is purely real in some domain, then it must be constant in that domain.

(2) Suppose we have a branch $f(z)$ of $z^{\frac{1}{2}}(i+z)^{\frac{1}{2}}$ analytic throughout the complex plane with a cut of $(x=0,-1\le y\le 0)$. If $f(\frac{1}{\sqrt{3}})=\sqrt{\frac{2}{3}}\,e^{-i\frac{5}{6}\pi}$, what is $f(i)$?【交大電信】

【參考解答】(1)設 $f(z)=u(x,y)+iv(x,y)$ 為解析，得知 u 必為一常數，即 $f(z)=u=c$，得證。

(2) $f(z)=z^{\frac{1}{2}}(z+i)^{\frac{1}{2}}$，分枝點 $z=0, z=-i$ 則

$$f(i)=\sqrt{2}e^{i(\frac{2n\pi+2m\pi-\pi}{2})}=\sqrt{2}e^{i[(n+m)\pi-\frac{\pi}{2}]}=\sqrt{2}e^{i(-\pi-\frac{\pi}{2})}=i\sqrt{2}$$

或 $f(i)=\sqrt{2}e^{i(\frac{\pi+2n\pi+2m\pi}{2})}=\sqrt{2}e^{i[(n+m)\pi-\frac{\pi}{2}]}=-\sqrt{2}i$

【例題2】

$$f(z)=\frac{1}{z^2+z+1}\ln\frac{z-i}{z+i}$$

(1) Identify all singularities.

(2) Using $z-i=r_1e^{i\theta_1}, z+i=r_2e^{i\theta_2}$, is $f(z)$ single valued?【91 成大微機電】

【參考解答】(1)令 $z^2+z+1=0$ 求奇異點，可得 $z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ 為一階極點。另外 $z=i,-i$ 為分枝點，且切割線上的每一點均為非孤立奇點。

(2)令 $z-i=r_1e^{i\theta_1}$，$z+i=r_2e^{i\theta_2}$，$z=r_1e^{i\theta}$，函數值改變，$f(z)$ 為多值函數，$z=i$ 為分枝點，同理 $z=-i$ 亦為分枝點。