提要 322：Cauchy-Riemann 方程式之極座標表示法

Cauchy-Riemann 方程式之極座標表示法
Cauchy-Riemann 方程式之極座標表示法為： $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ 、 $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$ 、 $r>0$

【證明】

已知 Cauchy-Riemann 方程式可表為：

$$\frac{\partial u(x,y)}{\partial x}=\frac{\partial v(x,y)}{\partial y} \text{ 、 } \frac{\partial u(x,y)}{\partial y}=-\frac{\partial v(x,y)}{\partial x} \tag{1}$$

另外，變數 (x,y) 與變數 (r,θ) 之關係為：

$$x=r\cos\theta \text{ 、 } y=r\sin\theta \tag{2}$$

所以：

$$\begin{aligned}\frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\\ \frac{\partial u}{\partial \theta}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)\end{aligned} \tag{3}$$

由式(3)知：

$$\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} \tag{4}$$

上式之反轉換可表為：

$$
\begin{aligned}
\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix}
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} \\
&= \frac{1}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} \\
&= \frac{1}{r\left(\cos^2\theta + \sin^2\theta\right)} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} \\
&= \frac{1}{r} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\dfrac{1}{r}\sin\theta \\ \sin\theta & \dfrac{1}{r}\cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta \dfrac{\partial u}{\partial r} - \dfrac{1}{r}\sin\theta \dfrac{\partial u}{\partial \theta} \\ \sin\theta \dfrac{\partial u}{\partial r} + \dfrac{1}{r}\cos\theta \dfrac{\partial u}{\partial \theta} \end{bmatrix}
\end{aligned} \tag{5}
$$

由式(5)知：

$$
\begin{cases} \dfrac{\partial u}{\partial x} = \cos\theta \dfrac{\partial u}{\partial r} - \dfrac{1}{r}\sin\theta \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial u}{\partial y} = \sin\theta \dfrac{\partial u}{\partial r} + \dfrac{1}{r}\cos\theta \dfrac{\partial u}{\partial \theta} \end{cases} \tag{6}
$$

根據式(6)，將符號u改寫為符號v，則：

$$
\begin{cases} \dfrac{\partial v}{\partial x} = \cos\theta \dfrac{\partial v}{\partial r} - \dfrac{1}{r}\sin\theta \dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial y} = \sin\theta \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\cos\theta \dfrac{\partial v}{\partial \theta} \end{cases} \tag{7}
$$

根據式(6)與式(7)，並由 Cauchy-Riemann 條件知，$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ 與 $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ 可分別改寫為：

$$\begin{cases} \cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial\theta}=\sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial\theta} \\ \sin\theta\dfrac{\partial u}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial u}{\partial\theta}=-\left(\cos\theta\dfrac{\partial v}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial v}{\partial\theta}\right) \end{cases} \tag{8}$$

上式中之$\sin\theta$與$\cos\theta$項次之係數應對應相等。基於此，由式(8)中之第一式可知：

$$\boxed{\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}\text{、}\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}} \tag{9}$$

由式(8)中之第二式亦可得知完全相同結果，故得證。