提要 298：兩個函數的 Jacobian 問題

兩個函數的 Jacobian 問題
已知 $f(x,y,u,v)=C_1$、$g(x,y,u,v)=C_2$，試推求 $\frac{\partial u}{\partial x}$、$\frac{\partial u}{\partial y}$、$\frac{\partial v}{\partial x}$、$\frac{\partial v}{\partial y}$。

解答：

首先對 $f(x,y,u,v)=C_1$、$g(x,y,u,v)=C_2$ 作全微分(Total Differential)之運算，亦即：

$$df(x,y,u,v)=dC_1 \tag{1a}$$

$$dg(x,y,u,v)=dC_2 \tag{1b}$$

上式可改寫為：

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=0 \tag{2a}$$

$$\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial u}du+\frac{\partial g}{\partial v}dv=0 \tag{2b}$$

由式(2a)與式(2b)可推求出：

$$\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \tag{3a}$$

$$\frac{\partial g}{\partial x}\frac{dx}{dx}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}=0 \tag{3b}$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial f}{\partial y}\frac{dy}{dy}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0 \tag{3c}$$

$$\frac{\partial g}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial g}{\partial y}\frac{dy}{dy}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial y}=0 \tag{3d}$$

由題意知，(u,v)是一組獨立變數、(x,y)是另一組獨立變數，故 $\frac{\partial y}{\partial x}=0$、$\frac{\partial x}{\partial y}=0$。基於此，式(3a)-(3d)亦可表為：

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \tag{4a}$$

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}=0 \tag{4b}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0 \tag{4c}$$

$$\frac{\partial g}{\partial y}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial y}=0 \tag{4d}$$

為輕便起見，各項微分式亦可分別表為$\frac{\partial f}{\partial x}=f_x$、$\frac{\partial f}{\partial y}=f_y$、$\frac{\partial f}{\partial u}=f_u$、$\frac{\partial f}{\partial v}=f_v$、$\frac{\partial g}{\partial x}=g_x$、$\frac{\partial g}{\partial y}=g_y$、$\frac{\partial g}{\partial u}=g_u$、$\frac{\partial g}{\partial v}=g_v$，則式(4a)-(4d)又可改寫為：

$$f_u\frac{\partial u}{\partial x}+f_v\frac{\partial v}{\partial x}=-f_x \tag{4a}$$

$$g_u\frac{\partial u}{\partial x}+g_v\frac{\partial v}{\partial x}=-g_x \tag{4b}$$

$$f_u\frac{\partial u}{\partial y}+f_v\frac{\partial v}{\partial y}=-f_y \tag{4c}$$

$$g_u\frac{\partial u}{\partial y}+g_v\frac{\partial v}{\partial y}=-g_y \tag{4d}$$

式(4a)-(4b)與式(4c)-(4d)可分別表為：

$$\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix}=\begin{bmatrix} -f_x \\ -g_x \end{bmatrix} \tag{5a}$$

$$\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{bmatrix}=\begin{bmatrix} -f_y \\ -g_y \end{bmatrix} \tag{5b}$$

應用 Cramer 定則解析式(5a)，即可推求出$\frac{\partial u}{\partial x}$、$\frac{\partial v}{\partial x}$之解：

$$\boxed{\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-f_x & f_v\\ -g_x & g_v\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{\begin{vmatrix}f_x & f_v\\ g_x & g_v\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{J\left(\dfrac{f,g}{x,v}\right)}{J\left(\dfrac{f,g}{u,v}\right)}} \qquad (6a)$$

$$\boxed{\frac{\partial v}{\partial x}=\frac{\begin{vmatrix}f_u & -f_x\\ g_u & -g_x\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{\begin{vmatrix}f_u & f_x\\ g_u & g_x\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{J\left(\dfrac{f,g}{u,x}\right)}{J\left(\dfrac{f,g}{u,v}\right)}} \qquad (6b)$$

同理，應用 Cramer 定則解析式(5b)，即可推求出 $\frac{\partial u}{\partial y}$ 、 $\frac{\partial v}{\partial y}$ 之解：

$$\boxed{\frac{\partial u}{\partial y}=\frac{\begin{vmatrix}-f_y & f_v\\ -g_y & g_v\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{\begin{vmatrix}f_y & f_v\\ g_y & g_v\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{J\left(\dfrac{f,g}{y,v}\right)}{J\left(\dfrac{f,g}{u,v}\right)}} \qquad (7a)$$

$$\boxed{\frac{\partial v}{\partial y}=\frac{\begin{vmatrix}f_u & -f_y\\ g_u & -g_y\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{\begin{vmatrix}f_u & f_y\\ g_u & g_y\end{vmatrix}}{\begin{vmatrix}f_u & f_v\\ g_u & g_v\end{vmatrix}}=-\frac{J\left(\dfrac{f,g}{u,y}\right)}{J\left(\dfrac{f,g}{u,v}\right)}} \qquad (7b)$$

附註：兩個函數 F 、 G 之 Jacobian 是定義為 $\boxed{J\left(\frac{F,G}{\alpha,\beta}\right)=\begin{vmatrix}F_\alpha & F_\beta\\ G_\alpha & G_\beta\end{vmatrix}}$ ，其中 $F_\alpha=\frac{\partial F}{\partial \alpha}$ 、 $F_\beta=\frac{\partial F}{\partial \beta}$ 、 $G_\alpha=\frac{\partial G}{\partial \alpha}$ 、 $G_\beta=\frac{\partial G}{\partial \beta}$ 。