提要 296：球體座標系統的 Laplacian

在 (x,y,z) 卡氏座標系統(Cartesian Coordinates)中，其 Laplacian ∇^2 係定義為 $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$。本單元旨在解釋 (r,θ,ϕ) 球體座標系統(Spherical Coordinates)下之 Laplacian 表示法。

球體座標系統的 Laplacian

如圖 1 所示球體座標系統 (r,θ,ϕ) 的 Laplacian ∇^2 可表為：

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

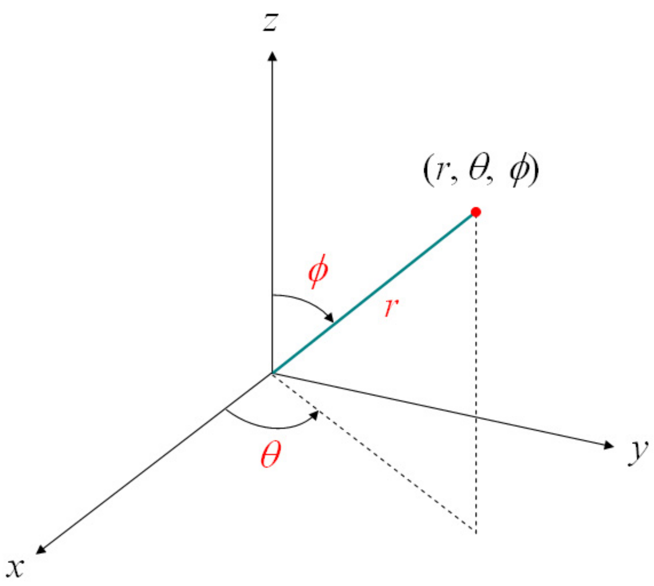

圖 1　球體座標系統 (r,θ,ϕ) 的表示法

證明：

觀察圖 1 知，卡氏座標系統(x, y, z)與球體座標系統(r,θ,ϕ)有以下之關係：

$$x = r\cos\theta\sin\phi \text{ 、 } y = r\sin\theta\sin\phi \text{ 、 } z = r\cos\phi \tag{1}$$

故

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r} = \cos\theta\sin\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\phi\frac{\partial}{\partial z} \tag{2a}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} \tag{2b}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \phi} = r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} - r\sin\phi\frac{\partial}{\partial z} \tag{2c}$$

式(2a)-(2c)可整理為：

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -r\sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ r\cos\theta\cos\phi & r\sin\theta\cos\phi & -r\sin\phi \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \tag{3}$$

上式之反轉換可表為：

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} &= \frac{1}{-r^2\sin\phi} \begin{bmatrix} -r^2\sin^2\phi\cos\theta & r\sin\theta & -r\sin\phi\cos\phi\cos\theta \\ r^2\sin^2\phi\sin\theta & -r\cos\theta & -r\sin\phi\cos\phi\sin\theta \\ -r\sin\phi\cos\phi & 0 & r\sin^2\phi \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix} \\ &= \begin{bmatrix} \sin\phi\cos\theta & -\frac{1}{r}\frac{\sin\theta}{\sin\phi} & \frac{1}{r}\cos\phi\cos\theta \\ -\sin\phi\sin\theta & \frac{1}{r}\frac{\cos\theta}{\sin\phi} & \frac{1}{r}\cos\phi\sin\theta \\ \frac{1}{r}\cos\phi & 0 & -\frac{\sin\phi}{r} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{bmatrix} \end{aligned} \tag{4}$$

亦即：

$$\frac{\partial}{\partial x}=\sin\phi\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\frac{\sin\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\phi} \tag{5a}$$

$$\frac{\partial}{\partial y}=-\sin\phi\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\frac{\cos\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\sin\theta\frac{\partial}{\partial\phi} \tag{5b}$$

$$\frac{\partial}{\partial z}=\frac{1}{r}\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi} \tag{5c}$$

所以：

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \\
&= \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)+\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right)+\left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z}\right) \\
&= \left(\sin\phi\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\frac{\sin\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\phi}\right)\left(\sin\phi\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\frac{\sin\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\phi}\right) \\
&+\left(-\sin\phi\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\frac{\cos\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\sin\theta\frac{\partial}{\partial\phi}\right)\left(-\sin\phi\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\frac{\cos\theta}{\sin\phi}\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\phi\sin\theta\frac{\partial}{\partial\phi}\right) \\
&+\left(\frac{1}{r}\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{r}\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) \\
&= \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\cot\phi}{r^2}\frac{\partial}{\partial\phi}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}
\end{aligned}$$

故得證。