交通大學

土木工程學系

91~97 學年度
工程數學考古題

國立交通大學九十一學年度碩士班入學考試試題

科目名稱：工程數學(151)　　考試日期：91 年 4 月 20 日 第 1 節

系所班別：土木工程學系　　組別：甲組　　第 1 頁,共 1 頁

*作答前,請先核對試題、答案卷（試卷）與准考證上之所組別與考試科目是否相符!!

1. 請以 power series 求解　(15%)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-9)y=0$$

2. 請以 Laplace Transform 求解　(20%)

$$\begin{bmatrix}50 & 0\\ 0 & 50\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+\begin{bmatrix}250 & -100\\ -100 & 250\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

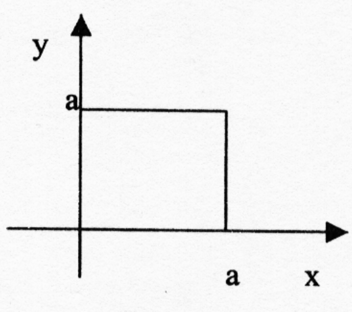
其起始條件為 $\begin{Bmatrix}x_1(0)\\ x_2(0)\end{Bmatrix}=\begin{Bmatrix}1.0\\ 0\end{Bmatrix}$, $\begin{Bmatrix}\dot{x}_1(0)\\ \dot{x}_2(0)\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$

3. 請利用向量函數（vector function, parametric equation or parametric representation）求得 circular cylinder：　(15%)

$x^2+y^2=a^2$,　$0\leqq z\leqq 2$ 表面（$x^2+y^2=a^2$ 所在之面）之單位正交向量（unit normal vector）

4. 請利用分離變數法求得下列問題之 steady-state solutions, u：　(20%)

控制方程式 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$



a = 2

u= $\sin\pi x$　on the upper side〔u(x,2)〕,

0　on the others。

5.已知 $\mathbf{K}=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$　(30%)

(a) 試求 **K** 之特徵值與特徵向量，特徵向量之正規化請依據向量長度為 1 的條件。

(b) 令前項特徵向量所組成之矩陣為 **X**，請以高斯消去法求 $\mathbf{X}^{-1}$。

(c) **X** 是否為正交矩陣(orthogonal matrix)？請根據正交矩陣之定義驗證之！

1. (30%) 試以級數解求解

$$x\frac{d^2y}{dx^2}-y=0;$$

並求該解之收斂半徑 。

2. (20%) 試求解

$$\frac{\partial^4 w}{\partial x^4}+k\frac{\partial^2 w}{\partial t^2}=0\text{，}k=0.01\text{，}\quad 0\le x\le 1\text{，}t\ge 0\text{；}$$

在 $x=0$ 與 $x=1$ 處，$w=\dfrac{\partial^2 w}{\partial x^2}=0$；當 $t=0$，$\dfrac{\partial w}{\partial t}=0$ 且 $w=\sin\pi x+0.5\sin 5\pi x$ 。

3. (25%) 試求下列線性系統之通解

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{pmatrix}\dfrac{d^2u_1(t)}{dt^2}\\ \dfrac{d^2u_2(t)}{dt^2}\end{pmatrix}+\begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix}\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

4. (15%) (a) 若運動方程式 $u''(t)+0.02u'(t)+25u(t)=p(t)$ ，且 $u(0)=u'(0)=0$ ，若定義其位移頻率響應函數 $u(i\omega)=H(i\omega)p(i\omega)$，其中 $u(i\omega)$ 爲 $u(t)$ 之傅立葉轉換， $p(i\omega)$ 爲 $p(t)$ 之傅立葉轉換，$i=\sqrt{-1}$。試求 $H(i\omega)$；

(10%) (b) 若已知 $h(t)=\mathcal{F}^{-1}\{H(i\omega)\}$，請將前述運動方程式之解 $u(t)$ 表示成 convolution integral 之形式。

1. 試求解微分方程 (30%)

$$m\ddot{x}+c\dot{x}+kx=\sin\omega t\,,\ t\geq 0$$

$$x(0)=1,\ \ \dot{x}(0)=0$$

其中 m,c,k,ω 均為常數；x 為 t 之函數。

2. 試求解微分方程 (20%)

$$y^{(iv)}=x^3+\delta(x-0.5),\ \ 0\leq x\leq 1$$

且 $y'(0)=y(0)=y(1)=y'(1)=0$；

其中 δ 為 Dirac delta function， y 為 x 之函數。

3. 已知 $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ 試求

(1) A 之特徵值與特徵向量 (15%)

(2) 令 X 為 A 之特徵向量所組成之特徵矩陣，求其逆矩陣 X^{-1} (10%)

(3) 求 A^{50} (10%)

4. 試利用 Laplace transform 證明 (15%)

$$\ddot{u}(t)+\omega_n{}^2u(t)=p(t)$$ 初始條件 $u(0)=\dot{u}(0)=0$

之解可表示成 $u(t)=\dfrac{1}{\omega_n}\int_0^t p(\tau)\sin[\omega_n(t-\tau)]d\tau$

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1. 若一兩自由度 mass-spring 系統之運動方程如下：

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix}+\begin{bmatrix}5 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\begin{Bmatrix}\sin\omega_1 t\\0\end{Bmatrix}$$

其中 ω_1 為該系統之最小自然振動頻率(rad/sec)。試求解其反應歷時函數(假設 zero initial conditions)。(15%)

2. 試求解兩階微分方程通解：(20%)

$y''+(1+x+x^2+2x^3)y'+3y=3x+5x^2$ ，where y is a function of x.

3. 試求解一階微分方程通解：(15%)

$2xyy'+(x-1)y^2=x^2e^x$ ，where y is a function of x.

4. 若矩陣 $A=\begin{bmatrix}1 & 0 & 3 & 2\\1 & 2 & 0 & 1\\2 & 6 & 3 & 0\\3 & 0 & 6 & 8\end{bmatrix}$, 試求：

(1) det(A)　(10%)

(2) rank A　(5%)

(3) A^{-1}　(10%)

5. 考慮一穩態流(steady flow), 其流速 $\vec{v}(t)=y(t)\vec{i}-4x(t)\vec{j}$

(1) 試求該流體在任意時間 t 之位置向量 $\vec{r}(t)$

(Hint: $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$; $\vec{v}(t)=\dfrac{d\vec{r}}{dt}$)　(10%)

(2) 若流體內某一分子在時間 $t=0$ 時之位置為(1,0), 當時間 $t=\dfrac{\pi}{4}$ 時它會到達什麼位置? (10%)

(3) 試說明該流體之流動軌跡. (5%)

1. (15%)Prove the convolution theorem of the Laplace transform.

2. (20%)Find the general solution for

$$x^3 y''' - 2x^2 y'' + 3xy' - 3y = 2x + 3x^3,$$

where y is a function of x.

3. (15%) Find the general solution for

$$y' = (y+x)(y+x-2) - 1$$

where y is a function of x.

4. (10%)Given a function $\Phi(x,y) = k(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)$, find the directional derivative of Φ along its boundary curve $C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5. (10%) Given a vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA$$

over the surface $S: \mathbf{r} = \begin{bmatrix} u\cos v & u\sin v & u^2 \end{bmatrix} 0 \le u \le 2, -\pi \le v \le \pi$ where $\mathbf{n}$ is the outer unit vector of S.

(Note: $\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F}[\mathbf{r}(u,v)] \cdot \mathbf{N}(u,v) du dv$ where $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$)

6. Given $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(a)Find the eigenvalues and eigenvectors of **A**. (10%)

(b)Let **P** be the eigen-matrix consisting of the eigenvectors of **A**, find $\mathbf{P}^{-1}$ using the method of Gauss-Jordan elimination. (10%)

(c)Show that the eigenvalues of the similarity matrix $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is the same as **A** and the eigenvectors of $\hat{\mathbf{A}}$ is $\mathbf{P}^{-1}\mathbf{x}$ where $\mathbf{x}$ is the eigenvector of **A**. (10%)

1. Solve the differential equation by using the Laplace transform: (25%)

$$y''+2y'+2y=r(t),\quad y(0)=1,\quad y'(0)=0$$

$$r(t)=\begin{cases} 0 & 0<t<\pi \\ 4\cos 2t & \pi<t<2\pi \\ 0 & 2\pi<t \end{cases}$$

2. Find a general solution of the following systems of ODEs by the method of undetermined coefficients: (25%)

$$y_1'=2y_1+2y_2+5e^{4t}$$

$$y_2'=5y_1-y_2-2e^{4t}$$

3. If λ's are eigenvalues of $\mathbf{A}$,
   (a) Show that the eigenvalues of $\mathbf{A}^{-1}$ are $1/\lambda$. (6%)
   (b) Show that the eigenvalues of $\mathbf{A}^2$ are λ^2. (6%)
   (c) If matrix $\mathbf{A}$ has characteristic determinant $\mathbf{D}(\lambda)=(\lambda-0.5)(\lambda-0.6)(\lambda-1)$, where λ is the eigenvalue of $\mathbf{A}$, what are the eigenvalues of $2\mathbf{A}^{-1}$. (8%)

4. Consider a steady flow flowing with velocity $\vec{v}(t)=-y(t)\vec{i}+x(t)\vec{j}$
   (a) Find the position vector, $\vec{r}(t)$, of the flow at any time t. (10%)
   (Hint: $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$; $\vec{v}(t)=\dfrac{d\vec{r}}{dt}$)
   (b) If some particle of the flow is initially (i.e. t = 0) at position (1,0), where will it be at time t = $\pi/2$? (5%)
   (c) What is the trajectory of the flow? (5%)
   (d) Is the flow incompressible? (5%)
   (e) Is the flow rotational? (5%)

1. $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 14 & -2 \\ -2.5 & 7.5 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} \sin\omega t \\ 0 \end{Bmatrix}$ with zero initial conditions, where a dot denotes the derivative with respect to time, t. What are the values of ω that make the solutions of $y_1(t)$ or $y_2(t)$ approach infinity as t reaches infinity? Find the solutions of $y_1(t)$ or $y_2(t)$ for such ω. (15%)

2. Solve $4xy'' + 2y' - y = 2x + x^2$ in terms of power series. (20%)

3. Please prove the following formulas or theorems:

(a) If $y_1(x)$ and $y_2(x)$ are two solutions of a homogeneous linear ODE (Ordinary Differential Equation), then a linear combination of these two solutions is still a solution of the homogeneous linear ODE. (7%)

(b) The Laplace transform of $\{f(t)/t\}$ is $\int_s^\infty F(\bar{s})d\bar{s}$, where $F(s)$ is the Laplace transform of $f(t)$. (8%)

4. A matrix $A=\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ is given.

(1) Find eigenvalues and their corresponding eigenvectors (10%)

(2) Find $\mathbf{X}^{-1}$ where $\mathbf{X}$ is the matrix of these eigenvectors. (10%)

(3) Find A^{10} through diagonalization of a matrix ($\mathbf{D}=\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$). (10%)

5. (a) Given a dynamic equation $u''(t)+\omega_0^{\ 2}u(t)=p(t)$ for $t\ge 0$, with initial conditions $u(0)=u'(0)=0$ where ω_0 represents the natural frequency of the system. If the frequency response is defined as $\hat{u}(\omega)=H(\omega)\hat{p}(\omega)$, where $\hat{u}(\omega)$ is the Fourier transform of $u(t)$ and $\hat{p}(\omega)$ is the Fourier transform $p(t)$. Please find $H(\omega)$； (10%)

(b) Denoting $h(t)=F^{-1}\{H(\omega)\}$， $t\ge 0$ (Note:You don't need to solve $h(t)$!)， Please find $u(t)$ in terms of convolution integral. (10%)

Note1： $F(f*g)=\sqrt{2\pi}F(f)F(g)$

$$(f*g)(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau=\int_{-\infty}^{\infty}f(t-\tau)g(\tau)d\tau$$

Note2： $F(f)=\hat{f}(\omega)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$

$$f(t)=F^{-1}\{\hat{f}(\omega)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}d\omega$$

交通大學

機械工程學系

91~97 學年度
工程數學考古題

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1. (12%) Solve

$$\frac{d^2y}{dx^2} + 4y = \cos 2x + \cos 4x$$

2. (11%)

(a) (6%) Find the following Laplace transform

$$L\{t^2 \cos \omega t\}$$

(b) (5%) Find the following inverse Laplace transform

$$L^{-1}\{\frac{6s-4}{s^2-4s-20}\}$$

3. (10%) Find the Fourier series of the following periodic function *f(t)*

$$f(t) = \begin{cases} 1+t^2 & ,0<t<1 \\ 3-t & ,1<t<2 \end{cases} \quad , f(t+2) = f(t)$$

4. (13%) Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$, where c is a constant.

5. (20%)

(a) (10%) Solve the linear system

$$\begin{aligned} 2x \quad +y \quad +2z \quad +w &= 6 \\ 6x \quad -6y \quad +6z \quad +12w &= 36 \\ 4x \quad +3y \quad +3z \quad -3w &= -1 \\ 2x \quad +2y \quad -z \quad +w &= 10 \end{aligned}$$

(b) (10%) Determine the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

6. (10%) Determine the radius of convergence of the following cases:

(a) $\sum_{n=0}^{\infty} n!x^n$,

(b) $\dfrac{1}{1-x}$,

(c) e^x.

7. (10%)

(a) Determine the all cube roots of 27.

(b) Evaluate $\int_C (1-z)dz$, where C is given by $z(t)=t-it$, t varying from 0 to 1.

8. (14%) The vibration of a stretched, flexible string problem:
An elastic string, stretched under a tension T between two points on the axis. The weight of the string per unit length after it is stretched we suppose to be a know function w(x). Besides the elastic and inertia forces inherent in the system, the string may also be acted upon by a distributed load $f(x,y,\dot{y},t)$. We assume that

1. The motion takes place entirely in one plan, and in this plan each particle moves at right angle to the equilibrium position of the string.
2. The deflection of the string during the motion is so small that the resulting change in the length of the string has no effect on the tension T.
3. The string is perfectly flexible, i.e., can transmit force only in the direction of its length.
4. The slope of deflection curve of the string is at all points and at all times so small.

(a) Derive the one dimension wave partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{Tg}{w(x)}\frac{\partial^2 y}{\partial x^2} + \frac{g}{w(x)} f(x,y,\dot{y},t),$$ where g is acceleration of gravity.

(b) Please check the dimensions (units) of three terms in (a).

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1. (10%) Solve $xy''+2y'-8x^3=0$, $y(1)=1$ and $y'(1)=2$.

2. (10%) Solve $y''''+8y''+16y=\cos x$.

3. (10%) Using Laplace transformation to solve
$y''+2y'+y=te^{-t}$, $y(0)=1$ and $y'(0)=-2$.

Formula: $$L(e^{-at}t^n)=\begin{cases}\dfrac{\Gamma(n+1)}{(s+a)^{n+1}} & ;n>-1\\ \dfrac{n!}{(s+a)^{n+1}} & ;n\ \ a\ \ positive\ \ integer\end{cases}$$

where gamma function $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dx$.

4. (10%) The arbitrary function f(t) of period 2p can be represented by a Fourier series of the form
$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{p}+b_n\sin\frac{n\pi t}{p}$$
Please obtain formula for the coefficients a_n, and b_n form this equation.

5. (15%) Wave partial differential equation with zero initial displacement states as follows:
$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2},\qquad (0<x<L,\ t>0)$$
$y(0, t) = y(L, t) = 0$　　$(t>0)$

$y(x, 0) = 0$　　$(0<x<L)$

$\frac{\partial y}{\partial t}(x,0)=1$　　$(0<x<L)$

Find the total solution of y(x, t).

6. (15%)

(a) Show that $\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})$ is in absolute value equal to the volume of a parallelepiped with sides **A**, **B** and **C** (**A**, **B** and **C** are vectors).

(b) If $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$, $\mathbf{B}=B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}$, $\mathbf{C}=C_1\mathbf{i}+C_2\mathbf{j}+C_3\mathbf{k}$ show that

$$\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

(c) Let $\mathbf{r}_1=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$, $\mathbf{r}_2=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$, $\mathbf{r}_3=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$ be the position vectors of points $\mathbf{P}_1(x_1,y_1,z_1)$, $\mathbf{P}_2(x_2,y_2,z_2)$ and $\mathbf{P}_3(x_3,y_3,z_3)$. Find an equation for the plane passing through $\mathbf{P}_1,\mathbf{P}_2$ and $\mathbf{P}_3$.

7. (15%)

(a) Find the value or values of k so that the following system of equation has solutions other than the trivial one.

$$\begin{cases} 4x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 - 11x_2 + kx_3 + 8x_4 = 0 \\ kx_2 - 4x_3 - 5x_4 = 0 \\ 2x_1 + 3x_2 - x_3 - 2x_4 = 0 \end{cases}$$

(b) Let $x_2=a$ and $x_4=b$ (a,b are any non-zero numbers), find x_1 and x_3.

8. (15%)

(a) Evaluate $\iint_R \sqrt{x^2+y^2}\,dxdy$, where R is the region in the xy plane bounded by $x^2+y^2=1$, and $x^2+y^2=9$.

(b) Evaluate $\iint_S \{xz^2dydz+(x^2y-z^3)dzdx+(2xy+y^2z)dxdy\}$, where S is the entire surface of the hemispherical region bounded by $z=\sqrt{1-x^2-y^2}$, and $z=0$. (15%)

國立交通大學 93 學年度碩士班入學考試試題

科目名稱：工程數學(171) 考試日期：93 年 4 月 17 日 第 1 節

系所班別：機械工程學系 組別：甲組 第 1 頁,共 2 頁

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1. (10%) Please find a real symmetric matrix C such that $Q = x^T Cx$, where Q equals $-3x_1^2 + 4x_1x_2 - x_2^2 + 2x_1x_3 - 5x_3^2$

2. (10%) Please find a parametric representation of the curve:
$x^2 + y^2 = 4$, $z = x^2$

3. (15%) Please find the eigenvalues and the eigenvectors of the matrix
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (10%) Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series with the period of 2π

5. (10%) Find the inverse Laplace transform of
$$F(S) = \frac{3s+7}{s^2 - 2s - 3}$$

6. (10%) Evaluate $\oint_C \frac{e^z}{\left(z^2+\pi^2\right)^2} dz$ where C is the circle $|z| = 4$; $z = x + yi$

7. (15%)
   (a) A particle P of mass 1g moves along the x-axis toward the origin O (initially x>0), in the same time, it is acted upon by a force equal to -4x. Determine the differential equation governing the motion of the particle.
   (b) Find the position of particle P as a function of time if the particle is initially at rest at x=10cm.
   (c) If the particle is also subject to a damping force that is numerically equal to four times the instantaneous velocity, find the position of the particle as a function of time.

8. (20%)

Solve the partial differential equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad (0<x<1,\ 0<y<1)$$

with the following conditions.

(a) $u(x,0)=0,\quad u(0,y)=0,\quad u(x,1)=1,\quad u(1,y)=0$

(b) $u(x,0)=x,\quad u(0,y)=0,\quad u(x,1)=0,\quad u(1,y)=\cos\left(\frac{\pi y}{2}\right).$

國立交通大學 94 學年度碩士班入學考試試題

科目名稱：工程數學(0011)　考試日期：94 年 4 月 17 日 第 1 節

系所班別：機械工程學系　　組別：甲組　　第 1 頁,共 1 頁

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1. (10%) Let $\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Find $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{A}^n$.

2. (10%) Find all the eigenvalues and eigenvectors of the matrix

$$\mathbf{A}=\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}$$

3. (15%) Evaluate $\iint_S [xz^2dydz+(x^2y-z^3)dzdx+(2xy+y^2z)dxdy]$ where S is the entire surface of the hemispherical region bounded by $z=\sqrt{1-x^2-y^2}$ and $z=0$.

4. (15%) Please find the general solution for the following Cauchy equation of the third order:

$$x^3y'''+3x^2y''-6xy'-6y=0$$

5. (20%) A function $u(x,y)$ satisfies the following equation, please find this function:

$$u_x+u_y=(ax^2+by)u$$

6. (10%) Solve $y''-y'-2y=4x^2$ with initial conditions $y(0)=0$; $y'(0)=5$

7. (10%) Find the inverse Laplace transform of

$$F(S)=\frac{s+3}{(s-2)(s+1)}$$

8. (10%) Write in the Fourie integeral form of $f(x)=\begin{cases}1, & |x|<1\\ 0, & |x|>1\end{cases}$

國立交通大學95學年度碩士班考試入學試題

科目：工程數學(3031)　　考試日期：95 年 3 月 11 日 第 1 節

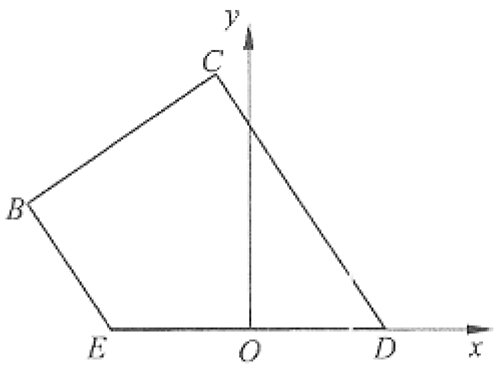
系所班別：機械工程學系　　組別：機械所丙所　　第 1 頁，共 2 頁

**作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!

1. Evaluate the integral

$$\int_A x\,dA \quad \text{and} \quad \int_A y\,dA$$

where A is the area of the quadrilateral $BCDE$ as shown. It is known that $EO = DO$, $\angle EBC = \angle BCD = 90^\circ$, $BC = 6$, $BE = 4$, $CD = 8$. (16%)



2. The initial value problem is given by

$$2x'' + 8(x - y) = 0$$

$$z = x - y$$

$$y(t) = \begin{cases} 2t^2, & 0 \le t \le \pi/6 \\ \dfrac{\pi^2}{18} + \dfrac{2\pi}{3}\left(t - \dfrac{\pi}{6}\right), & \pi/6 < t \end{cases}$$

$$x(0) = 2, \quad x_1'(0) = 0,$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$ and their first derivatives are continuous functions of t. Determine $z(t)$ for $t \ge 0$ and evaluate $z(\pi/6)$, $z'(\pi/6)$, $z(\pi/3)$ and $z'(\pi/3)$. (17%)

3. The governing equation for the vibration of a rectangular membrane with area $a \times b$ and mass density per unit area, ρ, subjected to tensile force per unit length, T, is expressed as

$$\partial^2 u/\partial^2 t = c^2 (\partial^2 u/\partial^2 x + \partial^2 u/\partial^2 y) \qquad \text{with} \qquad c^2 = T/\rho$$

$$0 \le x \le a, \; 0 \le y \le b$$

where t = time; u = u (x, y, t) = response of membrane.

(a) If the above equation satisfies the boundary condition, u = 0, on the boundary of the membrane for all $t \ge 0$, derive the expressions for determining the natural frequencies (eigenvalues) and mode shapes (eigenfunctions) of the membrane. (9%)

(b) If $a = 1$ and $b = 2$, plot the first five lower mode shapes of the membrane by showing the nodal lines on the graphs. (4 %)

(c) Discuss how the values of T and ρ affect the natural frequencies and mode shapes of the membrane. (4%)

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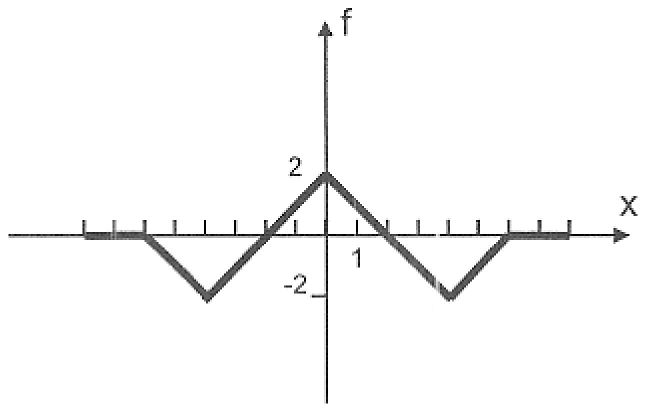
4. Find the Fourier series of the periodic function f(x) shown. (17%)



5. The *generalized eigenvalue problem* is given by $\mathbf{A}\,\mathbf{x} = \lambda \mathbf{B}\,\mathbf{x}$ if $\mathbf{B} \neq 0$. The eigenvalues λ_i satisfy the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{B}) = 0$, and the eigenvectors $\mathbf{x}_i$ are the nontrivial solutions of $(\mathbf{A} - \lambda \mathbf{B})\,\mathbf{x} = 0$.

(a) If matrices $\mathbf{A}, \mathbf{B}$ are symmetric, show that the eigenvectors satisfy the *generalized orthogonality* relations

$$\mathbf{x}_i^T \mathbf{A}\,\mathbf{x}_j = 0, \quad \mathbf{x}_i^T \mathbf{B}\,\mathbf{x}_j = 0, \quad \text{if } i \neq j. \qquad (6\%)$$

(b) If $\mathbf{A}, \mathbf{B}$ are symmetric 2×2 matrices, the modal matrix is defined as $\mathbf{U} = [\mathbf{x}_1 \ \ \mathbf{x}_2]$, where $\mathbf{x}_1, \mathbf{x}_2$ are the eigenvectors. The orthogonality results in the relation

$$\mathbf{U}^T \mathbf{A}\,\mathbf{U} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{U}^T \mathbf{B}\,\mathbf{U}.$$

Let the matrix $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$, two eigenvalues $\lambda_1 = 1$, $\lambda_2 = \dfrac{11}{6}$, and the first eigenvector $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. Determine the matrix $\mathbf{B}$. (10%)

6. Given a vector $\mathbf{F} = x(x^2\,\mathbf{i} - y^2\,\mathbf{j} + z^2\,\mathbf{k})$.

(a) Show $\int_C \mathbf{F} \cdot d\mathbf{R}$ is dependent on the integral path C which is a piecewise-smooth curve connecting two arbitrary points in xyz space. (8%)

(b) Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is the unit outer normal vector on the surface S bounding the circular cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 2$, including top and bottom surfaces. (9%)

國立交通大學96學年度碩士班考試入學試題

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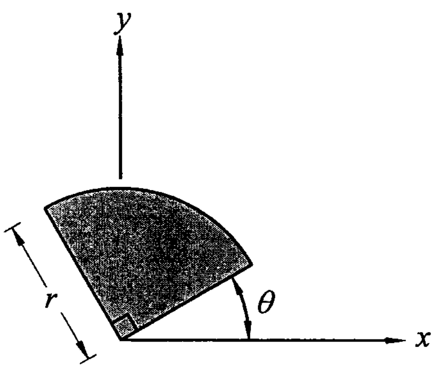
＊＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！　【可使用計算機】

1. Evaluate the integral

$$\int_A x^2 dA,\ \int_A y^2 dA \ \text{ and } \int_A xy dA$$

where A is the area of a quarter circle as shown. (16%)



2. The boundary value problem is given by

$$100\frac{d^2v}{dx^2} - v = C$$

$$v(0) = v(10) = 0,\ \ v'(0) = -10^{-2}$$

where $v = v(x)$, $v' = \dfrac{dv}{dx}$, and C is a constant to be determined.
Determine v, C, $v'(5)$ and $v'(10)$. (17%)

3. The two-dimensional wave equation of a vibrating membrane in the Cartesian coordinate system is expressed as

$$\partial^2 u/\partial t^2 = c^2 (\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$$

where u = u(x,y,t) and c is a constant.
(i). Use polar coordinates (r, θ) defined by x = r cosθ and y = r sinθ to derive the wave equation for the membrane.
(ii). For a circular membrane of radius R subjected to radially symmetric vibration, show that using the method of separating variables, the solution of the wave equation in polar coordinates can be converted to the solutions of two uncoupled ordinary differential equations.

(17%)

4. Let f(z) is the function defined by

$$f(z)=\frac{(z+1)}{(z+2)(z+3)}, \qquad \text{where } |z|<1,$$

which is computed using a power series expansion in the form of

$$f(z)=\sum_{n=0}^{\infty} f_n z^n$$

Please find f_n? (17%)

5. Consider the partial differential equation

$$\frac{\partial^4 w}{\partial x^4}+\frac{1}{\alpha^2}\frac{\partial^2 w}{\partial t^2}=0, \qquad 0<x<L, \quad t>0$$

subject to boundary conditions

$$w=\frac{\partial^2 w}{\partial x^2}=0 \quad \text{at } x=0; \qquad \frac{\partial^2 w}{\partial x^2}=\frac{\partial^3 w}{\partial x^3}=0 \quad \text{at } x=L.$$

Assume that the real-value solution is of the form $w(x,t)=F(x)\cos(\omega t)$.

(a) Derive the eigenfunctions $F(x)$, $0<x<L$, and the corresponding characteristic equation that ω must satisfy. Do not solve the characteristic equation. (12%)

(b) Determine the degenerate solution $F(x)$, $0<x<L$, for $\omega \to 0$. (5%)

6. Define a scalar function $\Phi(n_1, n_2, n_3)$ in the form

$$\Phi(n_1,n_2,n_3)=\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\, n_i n_j-\lambda(n_1^2+n_2^2+n_3^2-1)$$

where $\mathbf{n}=n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3$ is a unit vector, a_{ij} are components of a 3×3 matrix $\mathbf{A}$.

(a) Show that seeking the extreme value of $\Phi(n_1,n_2,n_3)$ yields an eigenvalue problem. (8%)

(b) Determine the eigenvalues λ_k $(k=1, 2, 3)$ and their corresponding eigenvectors $\mathbf{n}^{(k)}$ if

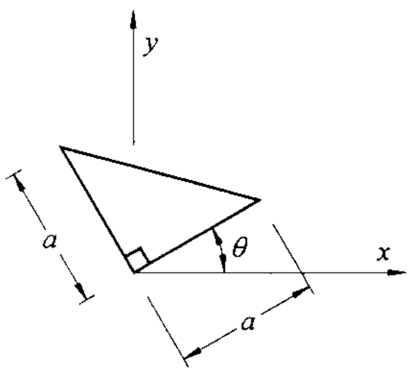
$$\mathbf{A}=\begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}. \qquad (8\%)$$

【可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Evaluate the integral

$$\int_A x^2 dA,\ \int_A y^2 dA \text{ and } \int_A xy dA$$

where A is the area of a right angle triangle as shown. (16%)



2. The system of differential equations is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix}$$

$$y_1(0) = 1,\ y_2(0) = -2,\ \dot{y}_1(0) = 2,\ \dot{y}_2(0) = 2$$

where $y_i = y_i(t)$, $\dot{y}_i = \dfrac{dy_i}{dt}$, $\ddot{y}_i = \dfrac{d^2 y_i}{dt^2}$, $i = 1, 2$.

Determine $y_i(t)$, $i = 1, 2$. (17%)

3. (a). Consider the following one dimensional wave equation:

$$U_{tt} = c^2 U_{xx}$$

with boundary conditions: U(0, t) = 0, U(L, t)= 0 for all t and initial conditions: U(x, 0) = f(x), $U_t(x, 0) = 0$.

The subscript $(\cdot)_t$ denotes partial derivative and c is wave speed.

Show that the solution of the above problem can be expressed as

$$U(x, t) = [f(x + ct) + f(x - ct)]/2 \qquad (10\%)$$

(b). If $f(x) = \sin(x\pi/L)$ for $0 \leqq x \leqq L$, plot the diagrams of f(x + ct), f(x − ct), and U(x, t) at t =L/2c and L/c. (7%)

4. Please find the centroid of a hemispherical volume $x^2 + y^2 + z^2 \le 1,\ z > 0$ (17%)

5. Consider the differential equation

$$\frac{d^4 y}{dx^4} + \alpha^2 \frac{d^2 y}{dx^2} = 0, \qquad 0 < x < L,\ \alpha > 0$$

subject to boundary conditions

$$y = \frac{dy}{dx} = 0 \qquad \text{at } x = 0,$$

$$y = \frac{d^2 y}{dx^2} = 0 \qquad \text{at } x = L.$$

(a) Fine the general solution $y(x)$. (8%)

(b) Derive the characteristic equation in terms of α and L. Do not solve it. (9%)

6. The scalar function $\phi(x_1, x_2, x_3)$ is continuous, with continuous first partial derivatives in the interior V of smooth closed surface S. Let the unit vector $\mathbf{n} = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3$ be outward normal to S, in which $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the base vectors of a Cartesian coordinate system.

(a) Show that

$$\int_V \frac{\partial \phi}{\partial x_j} dV = \int_S \phi\, n_j\, dS, \qquad j = 1, 2, 3.$$

Note that it is not the Gauss theorem. (8%)

(b) Show that

$$\int_S x_i n_j\, dS = \begin{cases} \bar{V}, & i = j \\ 0, & i \ne j \end{cases}$$

where $\bar{V}$ is the volume enclosed by surface S. (8%)