台灣大學

土木工程學系暨研究所

91~97 學年度
工程數學考古題

1. (15%) A linear transformation which maps a vector $\mathbf{x}$ to another vector $\mathbf{y}$ may be represented by $\mathbf{y} = A\mathbf{x}$, where both $\mathbf{x}$ and $\mathbf{y}$ are $n \times 1$ real matrices and A is an $n \times n$ real matrix.
   (a) (6%) What property does the matrix A must have for the (Euclidean) norms of $\mathbf{x}$ and $\mathbf{y}$ to be equal, *i.e.*, $\|\mathbf{x}\| = \|\mathbf{y}\|$. Why? Give a geometrical interpretation on the linear transformation.
   (b) (2%) What common property do the eigenvalues of the matrix A have in order for $\|\mathbf{x}\| = \|\mathbf{y}\|$?
   (c) (2%) What property may the eigenvector matrix of the matrix A have in order for $\|\mathbf{x}\| = \|\mathbf{y}\|$?
   (d) (5%) If the matrices A, $\mathbf{x}$ and $\mathbf{y}$ are complex instead of real and if $\|\mathbf{x}\| = \|\mathbf{y}\|$, then what are A, the eigenvalues of A, and the eigenvector matrix of A?

2. (18%) Let the vector field
$$\mathbf{F}(x, y, z) = \frac{x\mathbf{i} - z\mathbf{j} + y\mathbf{k}}{x^2 + y^2 + z^2},$$
the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and the line paths C be on the plane $x = 0$ and extend from the point $(0, 1, 0)$ to the point $(0, -2, 0)$. Are the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_C \mathbf{F} \times d\mathbf{r}$ independent of path? Why? Evaluate the line integrals.

3. (18%) Solve
$$\frac{dy}{dx} = \frac{ax + y - 2}{3y - 2}$$
for $y(x)$ with a being a real number. Note that different values of a may lead to different solutions, so that your solutions must include **ALL** possibilities.

4. (15%) For each set of the following five sets of boundary conditions,
   (a) $\phi(0, y) = 0,\ \phi_x(1, y) = 1,\ \phi(x, 0) = 0,\ \phi_y(x, 1) = 1;$
   (b) $\phi(0, y) = 0,\ \phi_y(0, y) = 1,\ \phi(x, 0) = 0,\ \phi_x(x, 0) = 1;$
   (c) $\phi(0, y) = 0,\ \phi_x(1, y) = 1,\ \phi(x, 0) = 1;$
   (d) $\phi(x, 0) = 0,\ \phi_y(x, 0) = 1;$
   (e) $\phi(0, y) = 0,\ \phi(1, y) = 1,\ \phi(x, 0) = 1,\ \phi_x(x, 0) = 0;$

   consider the applicability of the three different types of PDEs shown below:
$$\text{(i)}\ \ \phi_{xx} + \phi_{yy} = 0, \qquad \text{(ii)}\ \ \phi_{xx} - \phi_{yy} = 0, \qquad \text{(iii)}\ \ \phi_x - \phi_{yy} = 0,$$
   from which choose the PDE that the set of boundary conditions can be applied correctly. Then give the domain defined by the set of boundary conditions. **Explain the reason why the boundary conditions can not be used along with the other PDEs.** *Hint*: Consider the number of conditions needed and the physics of each equation.

5. (17%)
   (a) (7%) Find the roots of $1 + z^4 = 0$ and the sum of the residues of $\exp(iz)/(1 + z^4)$ in the upper half plane only.
   (b) (10%) Evaluate the integral $\int_0^\infty \frac{\cos x}{1+x^4} dx$ by the contour integral and the Cauchy integral theorem.

6. (17%) Given $v(y(x)) = \int_1^a \left(\frac{dy}{dx}\right)^2 x^3 dx$ and one fixed boundary condition $y(1) = 0$, find the extremal(s) for
   (a) (7%) $a = 2$ and $y(2) = 3$;
   (b) (10%) $1 < a < \infty$ and $y(a)$ lies on the curve $y = \frac{2}{x^2} - 3$.

試題隨卷繳回

1. (9%) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be arbitray vectors in three-dimensional (Euclidean) space. They may be linearly independent or may be linearly dependent.

   (a) Are $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ linearly independent? Why? Does your answer depend upon whether or not $\mathbf{a}$ and $\mathbf{b}$ are linearly independent?

   (b) Are $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$, and $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$ linearly independent? Why? (Prove your answer.) Does your answer depend upon whether or not $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly independent?

2. (12%) Let

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

   and $Q(x_1, x_2, x_3, x_4) = 4x_1^2 - 2x_1x_2 + 4x_2^2 + 4x_3^2 - 2x_3x_4 + 4x_4^2$.

   (a) Find the eigenvalues and normalized eigenvectors of A.

   (b) Discuss whether or not the normalized eigenvectors can be uniquely determined.

   (c) Is $Q(x_1, x_2, x_3, x_4)$ always positive or negative for any real numbers x_1, x_2, x_3, x_4? Why? (Prove your answer.)

3. (12%) Let $f(x) = \cos \pi x$ be a real-valued function defined **only** on the unit interval, $0 \le x \le 1$.

   (a) Find the Fourier series representation of $f(x)$.

   (b) Find the Fourier sine series representation of $f(x)$.

   (c) Which one of the above two representaions does give a better evaluation for $f(x)$ at $x = 0$? Why?

   (d) Which one of the above two representaions does give a better evaluation for $\frac{df}{dx}(x)$ at $x = 0$? Why?

4. (15%) Solve for $y(x)$ the following initial value problem,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-x},$$
$$y = 1 \text{ at } x = 0,$$
$$\frac{dy}{dx} = 0 \text{ at } x = 0.$$

5. (18%) Solve the following boundary value problem,

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0,$$
$$\frac{\partial f}{\partial r} = 2\cos\theta \text{ for } r = 2,$$
$$f = 3r\cos\theta \text{ as } r \to \infty,$$

   for the function $f(r, \theta)$ defined in the region $2 \le r < \infty, 0 \le \theta < 2\pi$ of a plane, for which (r, θ) is the polar coordinates.

6. (23%) Let z, z_0 be complex variables and $f(z)$ be a complex function.

   (a) (15%) Evaluate the integral $\int_C (z - z_0)^n dz$, (n = integer), along the circle C with center at z_0 and radius r described in the counterclockwise direction.

   (b) (8%) Find $\int_C f(z)dz$ if $f(z) = k$ (a constant), z, $\frac{1}{z}$, $\frac{2\sinh^2 z + 3\cosh 3z}{z}$, respectively, where C is any simple closed contour having $z_0 = 0$ in its interior, and C is taken in the positive direction.

7. (11%) Find the extremals for the following functionals:

   (a) $v(y(x)) = \int_2^3 y^2(1 - \frac{dy}{dx})^2 dx$ with $y(2) = 1$ and $y(3) = 3$;

   (b) $v(y(x), z(x)) = \int_0^1 y'z'dx$ with $y(0) = 0$, $y'(0) = 1$, $z(0) = 0$, and $z'(0) = 1$.

1. (17%) Let

$$\mathbf{A} = \begin{bmatrix} 89 & 48 & 0 \\ b & 61 & 0 \\ c & 0 & 1 \end{bmatrix}.$$

(a) Find the values of b and c such that the eigenvalues of $\mathbf{A}$ are all real and the eigenvectors are orthogonal to each other.

(b) For such b and c find the eigenvalues of A and the corresponding orthogonal diagonalizing matrix P and also its inverse matrix P^{-1}.

2. (16%) Let real-valued functions $g(x)$ and $h(x)$ defined over $0 \le x \le 1$ be expanded as

$$g(x) = \sum_{n=-\infty}^{\infty} G_n e_n(x), \quad h(x) = \sum_{n=-\infty}^{\infty} H_n e_n(x),$$

where $e_0(x) := 1$, $e_n(x) := \sqrt{2}\cos(2n\pi x)$, $e_{-n}(x) := \sqrt{2}\sin(2n\pi x)$ for $n = 1, 2, 3, \ldots$, and the coefficients G_n and H_n $(n = \ldots, -2, -1, 0, 1, 2, \ldots)$ are real numbers.

(a) Show that $\{\ldots, e_{-2}(x), e_{-1}(x), e_0(x), e_1(x), e_2(x), \ldots\}$ are a set of orthonormal functions defined over $0 \le x \le 1$.

(b) Find a formula relating $\int_0^1 g(x)h(x)dx$ to $\sum_{-\infty}^{\infty} G_n H_n$ and show clearly how to derive the formula.

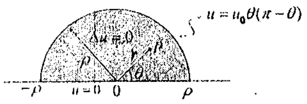
3. (18%) Solve the following initial value problems:

(a) (8%) $\frac{dy}{dx} - 2y^2 + 3y = 1$, $y(0) = 1$;

(b) (6%) $\frac{dy}{dx} = \frac{x+4y}{x}$, $y(0) = 0$;

(c) (4%) $\frac{dy}{dx} = \frac{x+4y}{x}$, $y(0) = 1$.

4. (15%) Find the steady state temperature distribution in the semicircular region of radius ρ lying in the upper half-plane and centered on the origin, as shown in the figure. The temperature on the straight boundary is $u = 0$, and that on the semicircular boundary is $u = u_0\theta(\phi - \theta)$.



5. (17%) Complex analysis.

(a) (7%) State the Cauchy Integral Formula and the Laurent Expansion Theorem.

(b) (10%) Find all the possible Laurent series about $z = 0$ for the complex-valued function $f(z) = \frac{1}{1-z}$, $(z = x + iy)$, by using the Laurent Expansion Theorem only.

6. (17%) Calculus of variations.

(a) (10%) Find the extremal for the functional $v(y(x)) = \int_0^1 \frac{\sqrt{1+(\frac{dy}{dx})^2}}{x}dx$ with boundary conditions $y(0) = 1$ and $y(1) = 0$.

(b) (7%) Find the transversality condition for the functional $v(y(x)) = \int_0^{x_1} \frac{\sqrt{1+(\frac{dy}{dx})^2}}{x}dx$ with boundary conditions $y(0) = 1$ and (x_1, y_1) constrained on a given curve $y_1 = \phi(x_1)$.

1. (20%) Let

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y}=\begin{bmatrix} y_1 \\ 0 \\ 3 \end{bmatrix}, \quad A=\begin{bmatrix} 89 & a_{12} & 0 \\ a_{21} & 61 & 0 \\ a_{31} & 0 & 1 \end{bmatrix}.$$

   (a) For the real vectors $\mathbf{x}$ and $\mathbf{y}$, are there some conditions which should be imposed upon x_1, x_2, x_3, y_1 such that $\mathbf{x}\times\mathbf{y}+\mathbf{y}\times\mathbf{x}=\mathbf{0}$? What conditions if there are? Why if there are not?

   (b) For the real matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution? What conditions if there are? Why if there are not?

   (c) For the real matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that A, A^2, A^3 are linearly depentent? What conditions if there are? Why if there are not?

   (d) For the complex matrix A, are there some conditions which should be imposed upon a_{12}, a_{21}, a_{31} such that A has real eigenvalues? What conditions if there are? Why if there are not?

2. (13%) State Green's theorem in the plane and show that the fundamental theorem of calculus $\int_a^b \frac{dF}{dx}dx = F(b)-F(a)$ for a function $F(x)$ defined over $a \le x \le b$ can be deemed as a special case of the Green theorem for functions $P(x,y)$ and $Q(x,y)$ defined over $a \le x \le b$ in the plane. Are there some conditions which should be imposed upon $F(x), P(x,y), Q(x,y)$ such that the theorems are valid? What conditions if there are? Why if there are not?

3. Find *all* solutions for the following differential equations:

   (a) (6%) $\frac{du}{dx}=\frac{u}{2x}$;

   (b) (5%) $\frac{d^2u}{dx^2}=\frac{1}{2x}\frac{du}{dx}$;

   (c) (11%) $x\frac{\partial u}{\partial x}=2y\frac{\partial u}{\partial y}$;

   (d) (11%) $\frac{\partial^2 u}{\partial x^2}=2\frac{\partial^2 u}{\partial y^2}$.

4. (24%) Complex analysis.

   (a) Locate all singularities for the following functions: (i) $f_1(z)=\text{Arg } z$, (ii) $f_2(z)=|z|$, (iii) $f_3(z)=\frac{z^3-2z^2}{3z^2+4z-i}$, (iv) $f_4(z)=\text{Re } z$, and determine which are isolated.

   (b) Find the branch points and branch cuts for the following functions: (i) $g_1(z)=\text{Log }(3-2z)$, (ii) $g_2(z)=\text{Log }(3z-2+4i)$, (iii) $g_3(z)=\text{Log}_{-\frac{\pi}{2}}\,(4-2z)$, (iv) $g_4(z)=\text{Log}_2\,(2z+1)$, and plot the required answers on the z planes.

5. (10%) Find the extremal for the functional $v(y(x))=\int y^2(1-\frac{dy}{dx})^2dx$ with $y(2)=1$ and $y(3)=3$, and plot the required extremal on the xy plane.

試題隨卷繳回

Problem 1. Consider matrix $\mathbf{A}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$ and column vector $\mathbf{x}=\begin{pmatrix}1\\2\\3\end{pmatrix}$.

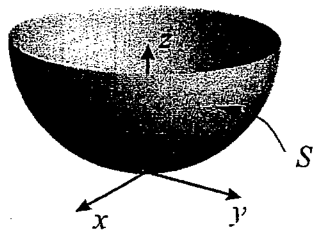
1) Find $\mathbf{y}=\mathbf{Ax}$. (5 %); **2)** Find $\mathbf{y}=\mathbf{A}^3\mathbf{x}$. (5 %); **3)** Find $\det(\mathbf{A}^9)$. (5 %)

4) Is matrix **A** an orthogonal matrix ? Show why or why not. (5 %)

5) Find a matrix **B** such that **x** is an eigenvector of **B**. (5 %)

Problem 2. With reference to the right figure, consider the surface S defined by

$$f(x,y,z)=x^2+y^2+(z-1)^2=1\text{; and } z\le 1$$



1) Find $\operatorname{grad} f=\nabla f$. (5 %)

2) Find unit normal vector $\mathbf{n}=(n_1,n_2,n_3)$ to surface S at position $(x,y,z)=(\frac{1}{3},\frac{2}{3},\frac{1}{3})$. (5 %)

3) Use Gauss's divergence theorem to determine the surface integral $\iint_S \mathbf{q}\cdot\mathbf{n}\,dA$

where **n** is the local unit normal to S and vector field **q** is defined as $\mathbf{q}=(q_1,q_2,q_3)=(2x,0,z)$. (15 %)

Problem 3. Given a partial differential equation: $\dfrac{\partial u(x,y)}{\partial x}+2\dfrac{\partial u(x,y)}{\partial x}=2u(x,y)+5\sin x$

1) Find the general solution of this PDE. (15%)

2) Provided that $u(x,0)=e^x$, find the exact solution of this PDE. (15%)

Problem 4. Given an ordinary differential equation: $y''(x)-y'(x)-2y(x)=36\cosh x$, with initial conditions $y(0)=3,\quad y'(0)=0$, find the solution. Note: <u>the complementary and the particular solutions must be individually specified.</u> (20%)

(Note: Reasonable conditions can be assumed by the examinee, provided that the conditions provided are insufficient.)

試題隨卷繳回

1. (20 %)
   (a) Find the Fourier series expansion, S_F, for $f(x) = -x,\ -1 < x < 1$.
   (b) Accordingly, find the Fourier series expansion, S_h, for $h(x) = 2 - x,\ 2 < x < 6$.

2. (20 %)
   For a vector function $\underline{F}(x,\ y,\ z) = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$, it is known that

$$curl\underline{F} = \nabla \times \underline{F} = (-4y^3z^6 - 4x^5y^2)\underline{i} - 4z^3\underline{j} + (20x^4y^2z - 3x^2y^2)\underline{k},$$
$$div\underline{F} = \nabla \bullet \underline{F} = 2xy^3 + 8x^5yz - 6y^4z^5.$$

   Find the possible F_1, F_2 and F_3. (Hint: There is no unique solution. The solution based on observation is recommended.)

3. (25 %)
   The definition of Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

   (a) Derive $L[\delta(t-c)] = e^{-sc}$, where $\delta(t)$ is the Dirac delta function.
   (b) Show $L[\frac{df(t)}{dt}] = sL[f(t)] - f(0)$, provided that $f(\infty) = 0$.
   (c) Show $L[f(t-a)H(t-a)] = e^{-as}L[f(t)]$, where $H(t)$ is the Heaviside step function.
   (d) Solve

$$\frac{d^2m(t)}{dt^2} = \delta(t-75),\quad 0 \le t < \infty,$$

   with

$$m(0) = m(\infty) = 0.$$

4. (10 %)
   The solution of the second order ordinary differential equation $y''(x) - 2y'(x) + y(x) = 0$ can be written as $y(x) = C_1y_1(x) + C_2y_2(x)$. If given $y_1(x) = e^x$, then derive the second solution as $y_2(x) = xe^x$.
   Note: no derivation, no score!

5. (25 %)
   For the second order ordinary differential equation

$$(2x+3)^2y''(x) - 2(2x+3)y'(x) + 4y(x) = 0,$$

   (a) derive and obtain the transformation $t = f(x)$ which transforms this differential equation to a constant coefficients second order ordinary differential equation of variable t (Hint: chain rule).
   (b) Write the transformed second order ordinary differential equation of variable t.
   (c) Solve $y(x) = ?$

試題隨卷繳回

Problem 1 (25%)

It is known that $A=\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix}$. If $\mathbf{B}=\mathbf{A}^6-2\mathbf{A}^5-3\mathbf{A}^4+9\mathbf{A}^3-4\mathbf{A}^2-6\mathbf{A}+8\mathbf{I}$, find **B** and $e^{\mathbf{B}}$, both solutions should be expressed in terms of a 3 x 3 matrix.

Problem 2 (25%)

The Fourier series representation of $g(x)=|x|$, for $-L \le x \le L$, has been found to be

$$g(x)=\frac{L}{2}-\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left[\frac{(2n-1)\pi x}{L}\right]}{(2n-1)^2}.$$

Given $f(x)=\begin{cases} 1 & \text{for } x<0 \\ -1 & \text{for } 0<x \end{cases}$ and $h(x)=\int_{-2\pi}^{x} f(t)\,dt$.

Find the Fourier series representation of $h(x)=\int_{-2\pi}^{x} f(t)\,dt$, for $-2\pi \le x \le 2\pi$.

Problem 3 (25%)

(a) Solve the following ordinary differential equation

$\frac{d^2\phi(x)}{dx^2}=1+x \quad -1\le x\le 1$, which is subjected to the boundary conditions:

$\phi(x=-1)=0$ and $\phi(x=1)=0$.

And identify the homogeneous, particular solutions and resonant modes.

(b) Express the solution of $\phi(x)$ from *Part(a)* in terms of the Chebyshev polynomials $T_n(x)$.

Hint: $T_0(x)=1$; $T_1(x)=x$; $T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0$; $(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0$;

$\int_{-1}^{1}\frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}dx=0,\ m\ne n$; $\|T_0(x)\|^2=\pi$; $\|T_n(x)\|^2=\pi/2$; $n=1,2,3,\ldots$.

Problem 4 (25%)

(a) Find the solution of the following partial differential equation (PDE):

$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,\quad r_1=3\le r\le r_2=5$

Where $u(r,\theta)$ is periodic in θ with period 2π and subject to the Dirichlet boundary conditions: $u(r_1=3,\theta)=F(\theta)=2+\cos\theta$; $u(r_2=5,\theta)=G(\theta)=1$

(b) Explain the applications of this PDE.

Hint: You may assume $u(r,\theta)=R(r)\,\Theta(\theta)$.

試題隨卷繳回

台灣大學

工程科學及海洋工程學系暨研究所

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1. Find the orthogonal trajectories of the curves: $y=\sqrt{x+c}$, where c is a constant. (15%)

2. Use the complex method to find the particular solution of the equation :
$m\ddot{y}+c\dot{y}+ky=F_o\cos(\omega t)$ (15%)

3. If the general solution of the equation $x^2y''+xy'+(x^2-\nu^2)y=0$
can be expressed by $y(x)=A\,J_\nu(x)+B\,J_{-\nu}(x)$
Fine the General solution (in terms of the Bessel function) of the equation:
$$x^2y''+(1-2\nu)xy'+\nu^2(x^{2\nu}+1-\nu^2)y=0$$
Hint: $u=x^{-\nu}y \quad z=x^\nu$ (20%)

4. Find the inverse matrix of **A** , where
$$\mathbf{A}=\begin{Bmatrix}0 & 0 & 1\\ 0 & 1/2 & 1\\ 1/3 & 0 & 0\end{Bmatrix}$$ (15%)

5. Consider the eigenvalue problem $\mathbf{Ax}=\lambda\mathbf{x}$, Show that the eigenvalues are real if A is a Hermitian matrix. (15%)

6. Consider a wave equation of $u(x,t)$,
$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} \qquad -\infty<x<\infty, \quad 0<t$$
if the initial conditions are given :
$$u(x,0)=\begin{cases}\cos(x) & when\ -\pi\le x\le\pi\\ 0 & otherwise\end{cases}$$
$$\frac{\partial u}{\partial t}(x,0)=0$$
Find and graph the waveform of $u(x,t)$ at $t=3.0$ (20%)

試題隨卷繳回

1. Solve the complete solution of the ordinary differential equation (10%)

$$y'' + k^2 y = f(x)$$

2. Find the value of k that makes the determinant zero (10%)

$$\begin{vmatrix} 1 & k & k+5 & k-3 & 100 \\ 0 & k & k-3 & k+5 & 100 \\ 0 & 0 & k+5 & k-3 & 100 \\ 0 & 0 & 0 & k-3 & k+5 \\ 0 & 0 & 0 & 0 & 100 \end{vmatrix} = 0$$

3. Evaluate the line integral $I = \oint_C zdx + xdy + ydz$ where C is the trace of the cylindrical surface: $x^2 + y^2 = 4$ and the plane $y + z = 4$. (hint: You may use Stoke's theorem.) (10%)

4. Find a matrix **P** such that $\mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{D}_\lambda$, where $\mathbf{D}_\lambda$ is a diagonal matrix formed by the eigenvalues of **A** (15%)

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

5. A tank is initially filled with 50 gal of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb of salt per gallon runs into the tank at the rate 5 gal/min, and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t and determine how long it will take for this amount to reach 75 lb. (15%)

6. Solve the partial differential equation (20%)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\phi(0, y) = 0, \quad \phi(a, y) = \sin(5\pi y/b)$$

$$\phi(x, 0) = 0, \quad \phi(x, b) = \sin(3\pi x/a)$$

7. Find the Fourier series of $f(x) = \dfrac{x^2}{4}$ $(-\pi < x < \pi)$, $f(x + 2\pi) = f(x)$

and also prove that $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots = \dfrac{\pi^2}{6}$ (20%)

試題隨卷繳回

1. Find the general solutions of the following differential equations of $y(x)$:

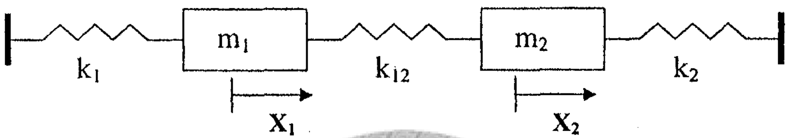
(i) $y' - 3y = x$

(ii) $\dfrac{d^4 y}{dx^4} - y = 0$

(iii) $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$

20%

2. Consider Mass-Spring system



If $m_1 = m_2 = k_1 = k_{12} = k_2 = 1$ and the initial conditions are

$$x_1(0) = x_2(0) = 1$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0$$

Find $x_1(t)$ and $x_2(t)$

20%

3. Given the Sturm-Liouville problem of $y(x)$:

$$y'' + \lambda^2 y = 0 \qquad 0 \le x \le L$$

$$y(0) = 0$$

$$y'(L) + Ay(L) = 0$$

Where L, A are constants. Find the eigenvalues and the normalized eigenfunction of the problem.

20%

4. Solve the integral equation of convolution type

$$y(t) = t^2 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$$

20%

5. Find the steady state solution of the following wave equation of $u(x,t)$: $\quad u_{tt} - c^2 u_{xx} = 0 \qquad 0 \le x \le L;\ 0 \le t$

the boundary conditions are:

$$u(0,t) = A\sin(\omega t)$$

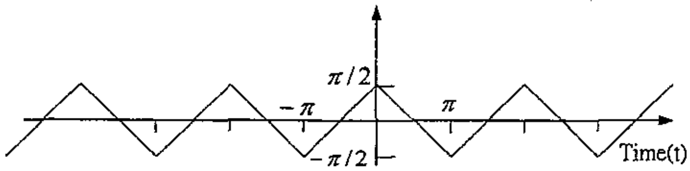
$$u(L,t) = 0$$

where c, A, ω are constants. What restrictions must be placed on ω ?

20%

1. Find the solution of the following equation: (15%)
$$y''+\lambda y=0,\quad y(0)=y(L),\quad y'(0)=y'(L)$$

2. A time signal distribution is shown in the figure. Find the frequencies and the corresponding amplitudes contained in this signal. (15%)

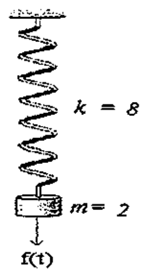


3. (a)Find the solution of the following wave equation (10%)
$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad -\infty<x<\infty,\quad 0<t<\infty,\quad ICs.\begin{cases}u(x,0)=f(x)\\ u_t(x,0)=g(x)\end{cases}$$
(b)If $f(x)=0,\ g(x)=x,\ 0\le x\le 1$, find the solutions of $u(-\frac{1}{2},\frac{1}{3})$, $u(2,5)$, and $u(\frac{1}{2},\frac{1}{6})$ (5%)

4. Evaluate the line integral of $\oint_C \vec{v}\cdot d\vec{r}$, where $\vec{v}=xz\,\vec{j}$, and C is the trace of surface $z=4-y^2$, cut off by the planes $x=0$, $z=0$ and $y=x$. (15%)

5. Find the principal axes and transform the following equation $2x_1^2+4x_1x_2+5x_2^2=1$ to its canonical form. (10%)

6. Consider a mass and spring vibration system as shown in the figure. The non-dimensional parameters are shown in the figure. Assume that the system is initially at rest. And at time t = 3, a unit impulse force is applied downward on the system suddenly, find the displacement of the system, (10%)



7. Find the principal value of $\displaystyle\int_{-\infty}^{\infty}\frac{1}{(x^2-7x+6)(x^2+4)}dx$ (10%)

8. A body of mass m is thrown vertically into the air with an initial velocity v_0. If the body encounters an air resistance proportional to its velocity, find (a) the velocity of the body at any time t and (b) the time at which the body reaches its maximum height.(10%)

試題隨卷繳回

(1) Find the eigen values and their corresponding eigen vectors of the following problem:
$\mathbf{Ax} = \lambda\mathbf{x}$

$$\text{where}\quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

15%

(2) Consider a problem of 3rd order O.D.E of $y(x)$: $y''' + y'' + y = \cos x$, if the initial conditions are given as: $y(0) = A; y'(0) = B; y''(0) = C$ where $A; B; C$ are constants. This problem can be rewritten by a system of three simultaneous 1st order O.D.E. of the form:

$$y_1' = f_1(y_1, y_2, y_3, x)$$
$$y_2' = f_2(y_1, y_2, y_3, x),$$
$$y_3' = f_3(y_1, y_2, y_3, x)$$

Find $f_1; f_2; f_3$ and the initial conditions of $y_1(x); y_2(x); y_3(x)$

15%

(3) Find the Fourier series expansion of the function $f(x)$: $f(x) = (1 + 2\sin 2x)^2$

15%

(4) The general solution of the 1-D free space wave equation of $u(x,t)$:

$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = 0$, is given by $u(x,t) = f(x - ct) + g(x + ct)$, where f and g are arbitrary functions of $(x - ct)$ and $(x + ct)$ respectively. Find the general solution of the following equation of $u(r,t)$:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0$$

15%

(5) Use the method of Laplace Transform, solve the following problem of $y(t)$:

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 2y = \delta(t) \quad ; \quad y(0) = \dot{y}(0) = 0$$

20%

(6) Determine an analytic complex function $f(z)$, where $z = x + iy$, such that $f(z) = u + iv$ and $u(x,y) = y^3 - 3x^2 y + y$.

20%

試題隨卷繳回

1. Find a family of curves which is the orthogonal trajectories of the curves: $y=\sqrt{x+c}$, where c is an arbitrary constant.
10%

2. The 2-D Laplacian operator in Cartesian coordinate is :
$\nabla^2 \equiv (\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$. Transform this operator in Polar coordinate: $x=r\cos\theta$; $y=r\sin\theta$
10%

3. Find the general solution of the following nonhomogeneous matrix equation:
$$\frac{d}{dt}\mathbf{y}=\mathbf{A}\mathbf{y}+\mathbf{g}$$
where $\mathbf{y}\equiv\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}$; $\mathbf{A}\equiv\begin{bmatrix} 2 & -4 \\ 1 & -3\end{bmatrix}$; $\mathbf{g}\equiv\begin{bmatrix} 10t \\ 9t+3\end{bmatrix}$
20%

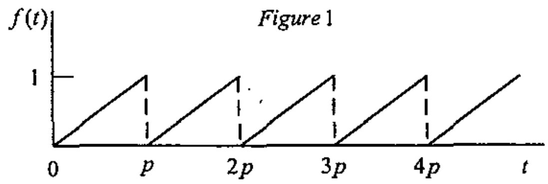
4. Consider the following two initial value problems of $y(t)$:
(I) $m\ddot{y}+c\dot{y}+ky=\delta(t)$ I.C. $y(0)=\dot{y}(0)=0$
(II) $m\ddot{y}+c\dot{y}+ky=r(t)$ I.C. $y(0)=\dot{y}(0)=0$
If $h(t)$ is the solution of problem (I), show that the solution of problem (II) can be expressed by:
$$y(t)=\int_0^t h(t-\tau)\cdot r(\tau)d\tau$$
20%

5. Consider the following eigenvalues problem of $y(x)$:
$y''+\lambda y=0 \quad a\le x\le b \quad ---(1)$
$B.C.\ \begin{cases} y(a)=y(b) \\ y'(a)=y'(b)\end{cases} \quad ---(2)$
if λ_m and λ_n are two distinct eigenvalues of the problem, show that the corresponding eigenfunctions $y_m(x)$ and $y_n(x)$ are orthogonal in (a,b).
20%

6. The equation of a spherical surface with radius "a" is given by : $x^2+y^2+z^2=a^2$. If the volume, V, of this closed surface is known : $V=\frac{4}{3}\pi a^3$, Use the Divergence theorem to find the surface area of this sphere.
Hint : Divergence Theorem : $\iiint_V \nabla\cdot\vec{F}dV=\iint_S \vec{F}\cdot\vec{n}dA$
20%

試題隨卷繳回

1. (15%) Find the Laplace transform of the following function $f(t)$ in Figure 1.



2. (15%) Find the eigenvalues and eigenfunctions of the differential equation $y''+\lambda y=0$ with the boundary conditions $y(0)=y(1), \quad y'(0)=y'(1)$

3. (15%) Solve the following equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)u(x,y)=\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{b}{2}\right)$$

BCs. $u(0,y)=u(a,y)=u(x,0)=u(x,b)=0$

4. (15%) A periodic function $f(x)=f(x+T)$ is approximated by the finite sum of its Fourier series $f(x)\approx P_k(x)\equiv A_0+\sum_{n=1}^{k}[A_n\cos n\omega_0 x+B_n\sin n\omega_0 x], \omega_0=\frac{2\pi}{T}$ and the total mean square error is defined as $E_k\equiv\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}[f(x)-P_k(x)]^2dx$. If the coefficients in $P_k(x)$ are determined by the Euler Formulae, prove that the approximation has the "least total mean square error" property.

5. (20%) Find a matrix **P** such that $\mathbf{P}^T\mathbf{A}\mathbf{P}=\mathbf{D}_\lambda$, where $\mathbf{D}_\lambda$ is a diagonal matrix formed by the eigenvalues of **A**

$$\mathbf{A}=\begin{bmatrix}9&1&1\\1&9&1\\1&1&9\end{bmatrix}$$

6. (20%) Find the integrals $\int_0^\infty\frac{(\ln x)^2}{2^2+x^2}dx$

試題隨卷繳回

台灣大學

機械工程學系暨研究所

91~97 學年度
工程數學考古題

1. Given a 3×3 matrix $\mathbf{A}$ as

$$\mathbf{A}=\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

(a) (5%) Find the rank of **A**.

(b) (5%) Find a basis for the column space of **A**.

(c) (5%) Find a basis for the row space of **A**.

(d) (5%) The null space of a matrix **A** consists of all vectors **x** such that $\mathbf{Ax}=0$. It is denoted by $N(\mathbf{A})$. Determine the rank of $N(\mathbf{A})$.

(e) (5%) Find a basis for the null space $N(\mathbf{A})$.

(f) (5%) Let $\mathbf{b}=[b_1,b_2,b_3]^T$. Under what conditions on **b** (if any) does $\mathbf{Ax}=\mathbf{b}$ have a solution?

(g) (5%) Determine $\det(\mathbf{A})$.

(h) (5%) Find the eigenvalues and eigenvectors of **A**.

2. (10%) Solve the ordinary differential equation below with $u(0)=u_0$ and find the steady state solution $u_\infty=\lim_{t\to\infty}u(t)$:

$$\frac{du}{dt}=au-bu^2,$$

where a, b are positive constants.

3. (15%) Determine the Fourier coefficients a_n, b_n of the Fourier series

$$a_o+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

representing the function $f(x)=\cos^2(x/6)-\sin(x/6)\cos(x/6)$ over the interval $[-3\pi,3\pi]$.

4. Consider the nonhomogeneous heat conduction equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+F(x,t)\quad \text{for}\quad 0<x<L,\quad t>0$$

subject to boundary conditions: $u(0,t)=u(L,t)=0$ for $t>0$,

and initial condition: $u(x,0)=f(x)$ for $0\le x\le L$.

(a) (10%) First, solve the above problem for the special case $F(x,t)=0$ (i.e. homogeneous case) by using the method of separation of variables.

In an attempt to solve the above nonhomogeneous equation, it is assumed that the solution can be written in the form:

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

(b) (5%) Derive the equation for the unknown function $T_n(t)$.

(c) (5%) Solve for $T_n(t)$ to get the complete solution $u(x,t)$.

5. Write down true (T) or false (F) to the following statements.

(a) (3%) If a vector function $\underline{v}$ satisfies $\nabla \times \underline{v} = 0$ everywhere in the region Ω, then $\oint \underline{v} \cdot d\underline{r} = 0$ for any closed curve C lying entirely within the region Ω.

(b) (3%) The complex function $f(z) = (z+1)^i$ is a multi-valued function.

(c) (3%) The complex integral $\oint_C \dfrac{z\,dz}{(z+1)(z-3i)} = 0$ over the closed curve C: $|z| = 1$.

(d) (3%) The complex integral $\oint_C e^{1/z^2}\,dz = 2\pi i$ over a closed curve C enclosing the origin.

(e) (3%) The complex function

$$\frac{1}{z(z^2+9)}$$

has a Taylor series expansion about the point $z = 2i$ valid in an annulus $0 < |z| < 3$.

試題隨卷繳回

1. (15%)

(1) By assuming $y(x) = ax^2 + bx + c$, find a particular solution of the following ordinary differential equation $\left(1 - x^2\right)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$.

(2) Find the general solution of the ordinary differential equation in (1).

2. (15%) Solve the following initial-valued problem: $\dfrac{d^2y}{dx^2} - y = 5\sin^2 x$; $y(0) = 2, y'(0) = -1$.

3. (20%) Answer the following questions:

(1) Let R be the set consisting of all real numbers, and S be the set consisting of all positive real numbers and zero. Is S a subspace of R? If not, why not?

(2) Are the following vectors in the four dimensional real vector space, R^4, linearly dependent: (1,3,-1,4), (3,8,-5,7), (2,9,4,23)? If yes, why? If not, why not?

(3) Consider a vector space V defined as the set consisting of all the solution functions $y(x)$ of the ordinary differential equation: $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$. Find a basis of the vector space V.

(4) Let A be a 3x3 matrix and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Find the nonsingular matrix P that diagonalizes A.

4. (40%) Choose the correct answer in each of the following questions. (4% each, derivations and reasoning are **not required**.)

(1) Which of the following equations represents a possible streamline of the vector field $\underline{\mathbf{F}} = (1/x)\underline{\mathbf{i}} + e^x\underline{\mathbf{j}}$?

(a) $\begin{cases} y = e^x + 1 \\ z = 0 \end{cases}$; (b) $\begin{cases} y = xe^x - e^x + 1 \\ z = 4 \end{cases}$; (c) $\begin{cases} y = (e^x/x) - 1 \\ z = 0 \end{cases}$; (d) $\begin{cases} x^2 = 1 \\ y = e^x \end{cases}$; (e) $\begin{cases} y = e^x \\ z = 4 \end{cases}$.

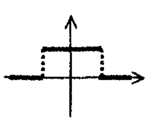
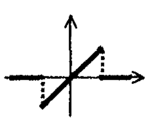
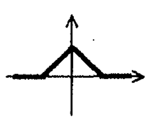
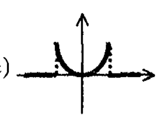
(2) What is the directional derivative of the function $\phi(x, y, z) = 8xy^2 - xz$ at the point (1,0,2) in the direction $\underline{\mathbf{u}} = \underline{\mathbf{i}} + \underline{\mathbf{j}} + \underline{\mathbf{k}}$?

(a) –3; (b) 3; (c) $-\sqrt{3}$; (d) $-1/\sqrt{3}$; (e) –1.

(3) Let $\phi(x, y, z) = -(x^2 + y^2 + z^2)^{-1/2}$. What is the flux of $\nabla\phi$ across the sphere of radius 3 centered at the origin?

(a) 4π; (b) 0; (c) 2π; (d) 36π; (e) -36π.

(4) If the Fourier transform of a function $f(t)$ is $\Im(\omega) = 2(1 - \cos\omega)/\omega^2$, then which of the followings is the most probable form of the function $f(t)$?

(a)  ; (b)  ; (c)  ; (d)  ; (e)  .

(5) Let $f(x) = 3x^2 - 1$ and let the Fourier series representation of $f(x)$ in the interval [-1,1] be $\sum_{n=0}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right]$, then, which of the following statements is true?

(a) $\sum_{n=0}^{\infty} b_n^2 = 1$; (b) $\sum_{n=0}^{\infty}(a_n^2 + b_n^2) = \dfrac{9}{5}$; (c) $\sum_{n=0}^{\infty}(a_n^2 + b_n^2) = \dfrac{18}{5}$; (d) $\sum_{n=0}^{\infty} a_n^2 = \dfrac{4}{5}$; (e) $\sum_{n=0}^{\infty} a_n^2 = \dfrac{8}{5}$.

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(6) If the Fourier transform of the function $f(t)=e^{-at^2}$ is $\Im(\omega)=\sqrt{\pi/a}\ e^{-\omega^2/4a}$, then, the Fourier transform of the function $g(t)=(t+2)\ e^{-a(t+2)^2}$ is:

(a) $-\sqrt{\pi/a}\ e^{-2i\omega}\cdot e^{-\omega^2/4a}$; (b) $-i\omega\sqrt{\pi/a}\ e^{-2i\omega}\cdot e^{-\omega^2/4a}$; (c) $\sqrt{\pi/a}\ e^{2i\omega}\cdot e^{-\omega^2/4a}$;

(d) $\dfrac{-i\omega}{2a}\sqrt{\pi/a}\ e^{2i\omega}\cdot e^{-\omega^2/4a}$; (e) $\dfrac{-i\omega}{2a}\sqrt{\pi/a}\ e^{-2i\omega}\cdot e^{-\omega^2/4a}$.

(7) What is the value of the complex integral $\oint_C (\bar{z}/z)\,dz$ where C is a closed circle of radius 2 about the origin?

(a) $2\pi i$; (b) 0; (c) -2π; (d) -4π; (e) $4\pi i$.

(8) The residue of the complex function $f(z)=\dfrac{\sin z}{z^2(z^2+i)}$ at $z=0$ is:

(a) $-i$; (b) i; (c) $-i/2$; (d) 0; (e) $\sqrt{-i}$.

(9) Let $f(z)=\dfrac{z+2}{(z-1)(z^2+4)}$, then which of the following statements is true?

(a) $f(z)$ has a double pole at $z=-2$; (b) $f(z)$ has a simple pole at $z=2$; (c) $f(z)$ has a simple zero at $z=1$; (d) $f(z)$ has a Taylor series expansion at $z=2$; (e) $f(z)$ is analytic in $0<|z-1|<\infty$.

(10) Let Ω be a 2-D simply connected domain with boundary curve C. If $\phi(x,y)$ satisfies the Laplace equation $\nabla^2\phi=0$ everywhere in Ω, then, which of the following statements is true?

(a) $\partial\phi/\partial y=-\partial\phi/\partial x$ in Ω; (b) ϕ has a local minimum (or maximum) in the interior of Ω;

(c) normal derivative $\partial\phi/\partial n=0$ on the curve C; (d) $\nabla\phi$ is in the direction normal to the curve C;

(e) $\oint_{\bar{C}}\nabla\phi\cdot d\underline{r}=0$ along any closed curve $\bar{C}$ lying within Ω.

5. (10%) By using the method of separation of variables, solve the following 2-D Laplace equation in polar coordinates (r,θ):

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\qquad 0\le r\le R,\ 0\le\theta\le 2\pi$$

subject to the boundary condition: $u(R,\theta)=1-\cos^2\theta$.

試題隨卷繳回

1. (30%) Write down on the answer sheet the correct answer to each of the following questions. (Derivations are not required.) **(本題請於答案卷之「選擇題作答區」內作答)**

(1) The directional derivative of the scalar function $\varphi(x,y,z)=xy-z^2$ evaluated at point $(1,-1,1)$ along the direction $3\,\underline{\mathbf{i}}-4\,\underline{\mathbf{k}}$ is

(a) 5 ; (b) 1 ; (c) -11/5 ; (d) 11/5 ; (e) 11.

(2) The line integral $\oint_C \underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ of the 2-D vector function $\underline{\mathbf{F}}=\frac{-y}{x^2+y^2}\,\underline{\mathbf{i}}+\frac{x}{x^2+y^2}\,\underline{\mathbf{j}}$ evaluated over the closed path C: $(x-2)^2+(y-2)^2=9$ is

(a) π; (b) 0 ; (c) 2π; (d) 6π; (e) $\pi/2$.

(3) Let $\underline{\mathbf{F}}(r,\theta)=(-1/r)\,\underline{\mathbf{e}}_r$ be a 2-D vector function given in terms of polar coordinates (r,θ) with $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_\theta$ denoting the base vectors of the coordinate system, then $\nabla\cdot\underline{\mathbf{F}}=?$

(a) 0 ; (b) $-1/r^2$; (c) $1/r^2$; (d) $1/r^3$; (e) $2/r^2$.

Let $f(x)=a_o+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$ be the Fourier series representation of the function $f(x)$ over the interval $-L\le x\le L$. Answer questions (4)~(6).

(4) Which of the following statements regarding to the above Fourier series is true?

(a) $a_o=\frac{1}{L}\int_{-L}^{L}f(x)\,dx$; (b) If $f(x)$ is an odd function in [-L, L], then $b_n=0$ for all n ;

(c) $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$; (d) $a_n=\frac{2}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx$;

(e) $\frac{1}{2L}\int_{-L}^{L}f^2(x)\,dx=a_o^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

(5) If in the interval [-2, 2], $f(x)$ is defined as $\begin{bmatrix} f(x)=5 & x=-2 \\ \quad =-x & -2<x\le 0 \\ \quad =x^2-1 & 0<x\le 2 \end{bmatrix}$, then the Fourier series at $x=2$ converges to

(a) 7/2 ; (b) 3 ; (c) 0 ; (d) 4 ; (e) 5/2.

(6) Let $f(x)=\cos^2(\pi x/2)$ in the interval [-2, 2], then which of the following statements is true?

(a) $a_n=0$ for all n ; (b) $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=1/4$; (c) $a_o=1/4$; (d) $a_1=1$; (e) $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=1/2$.

(7) Let $\delta(t)$ denote Dirac delta function and $f(t)=\cos t$, then the Fourier transform of $\delta(t-2)f(t)$ is :

(note that Fourier transform is defined as $\Im\{g(t)\}=\int_{-\infty}^{\infty}g(t)e^{-i\omega t}dt$)

(a) 0 ; (b) $e^{-2i\omega}$; (c) $e^{-i(\omega+2)}/\omega$; (d) $e^{-2i\omega}\cos 2$; (e) 1.

(8) Let $z=x+iy$ denote complex variable, then which of the following statements is true?

(a) $f(z)=\ln z$ is an analytic function in $-2\pi\le\arg(z)\le 2\pi$; (b) $z^3+i=0$ has infinitely many complex roots; (c) $f(z)=\sqrt{z}$ has a Taylor series expansion about $z=0$; (d) $f(z)=(1-\cos z)/z$ has a simple pole at $z=0$; (e) $\oint_C dz/[z(z^2+4)]=0$ over the closed circle C: $|z+3|=2$.

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(9) The residue of the complex function $f(z)=\sin z/[z(z+i)^2]$ at $z=-i$ is
(a) $i\cos i-\sin i$; (b) $-i\sin i$; (c) $-i\cos i+\sin i$; (d) 0; (e) $(i\cos i-\sin i)/2$.

(10) What is the value of the complex integral $\oint_C ze^{1/z}dz$ over C: $|z|=2$?
(a) 0; (b) $2\pi i$; (c) $4\pi i$; (d) πi; (e) 2π.

2. (10%) Consider the following equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-u$$

subject to boundary conditions: $u(0,t)=u(1,t)=0$ $(t>0)$
and initial condition: $u(x,0)=\sin(\pi x)\cos(\pi x)$ $(0\le x\le 1)$
Solve the problem by using the method of separation of variables. **<u>(Other methods are not allowed.)</u>**

3. (15%) Find the positive eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem:
$y''+(1+\lambda)y=0$, $y(0)+y'(0)=0$, $y(\pi)+y'(\pi)=0$

4. (15%) Find the general solution of $\dfrac{dy}{dx}=\dfrac{2x^2-y}{x\ln(x)}$.

5. (15%) Answer the following questions.
(1) Consider a set V, consisting of all the real solution functions $y(x)$ of the ordinary differential equation: $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9y=0$. Is V a real linear vector space? If yes, find the dimension and a basis of the vector space V.
(2) The vector v has components (1,-2,-1) with respect to the basis {(1,-1,1),(1,1,0),(1,0,1)} of R^3. Find its components with respect to the standard basis {(1,0,0),(0,1,0),(0,0,1)}.
(3) Which set or sets of the following vectors can form a basis for R^3? (a), (b), (c), and/or (d)?
(a) (1,2,-1) and (0,3,1)
(b) (2,4,-3), (0,1,1), and (0,1,-1)
(c) (1,5,-6), (2,1,8), (3,-1,4), and (2,1,1)
(d) (1,3,-4), (1,4,-3), and (2,3,-11)

6. (15%) Consider the following system of ordinary differential equations

$$\frac{du}{dt}=Au \text{ where } u=\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix}\in R^3 \text{ and } A=\begin{pmatrix}1&-1&0\\0&0&1\\1&-1&1\end{pmatrix}.$$

(1) Find the eigenvalues and the associated eigenvectors of matrix A.
(2) Find the exponential of the matrix At.
(3) Find the general solution of the system of ordinary differential equations.

試題隨卷繳回

1. (30%) Write down on the answer sheet the correct answer to each of the following questions. (Derivations are not required.)

(1) The line integral $\oint_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ of the 2-D vector function $\underline{\mathbf{F}} = (\frac{x^2+y^2}{2} + 2y)\,\underline{\mathbf{i}} + (xy - ye^y)\,\underline{\mathbf{j}}$ evaluated over the closed path C defined by : $\begin{cases} y = \pm 1, & -1 \le x \le 1 \\ x = \pm 1, & -1 \le y \le 1 \end{cases}$ is

(a) 4 ; (b) -8 ; (c) 0 ; (d) $-(e^1 - e^{-1})$; (e) $2(e^1 - e^{-1})$.

(2) Let $\underline{\mathbf{F}}(r,\theta) = (-1/r)\,\underline{\mathbf{e}}_r$ be a 2-D vector function given in terms of polar coordinates (r,θ) with $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_\theta$ denoting the base vectors of the coordinate system. What is the flux of $\underline{\mathbf{F}}$ across the circle of radius 2 centered at the origin?

(a) 0 ; (b) -2π ; (c) -4π ; (d) 4π ; (e) $\pi/2$.

(3) Let ϕ and $\underline{\mathbf{F}}$ be continuous and differentiable scalar and vector functions, respectively; then $\nabla \times (\phi \underline{\mathbf{F}}) = ?$

(a) $(\nabla\phi) \times \underline{\mathbf{F}} + \phi \nabla \underline{\mathbf{F}}$; (b) $(\nabla\phi) \cdot \underline{\mathbf{F}} + \phi(\nabla \times \underline{\mathbf{F}})$; (c) 0 ; (d) $(\nabla\phi)\underline{\mathbf{F}} + \phi(\nabla \times \underline{\mathbf{F}})$; (e) $\phi\nabla \times \underline{\mathbf{F}} - \underline{\mathbf{F}} \times (\nabla\phi)$.

(4) Let $z = x + iy$ denote the complex variable and $f(z)$ a complex function, then which of the following statements is true?

(a) $f(z) = 1/\sqrt{z}$ is an analytic function on the whole z-plane excluding the origin;

(b) $f(z) = z\cos(1/z)$ has a simple pole at $z = 0$;

(c) $f(z) = (z+2i)/[z(z^2+4)]$ has a Taylor series expansion about $z = -2i$;

(d) $z^{2/3} = -1$ has only two roots $z = \pm i$;

(e) $\oint_C \frac{(z-i)}{(z+3)(z^2+4)} dz = 0$ over the closed circle C: $|z-i| = 1$.

(5) The residue of the complex function $f(z) = (z+2)e^{1/z}$ at $z = 0$ is

(a) 5/2 ; (b) 1 ; (c) 1/2 ; (d) 0 ; (e) πi.

(6) What is the value of the complex integral $\oint_C \{(\sin z)/[z(z-i)^2]\}dz$ over C : $|z-i| = 2$?

(a) $\sin i - i\cos i$; (b) 0 ; (c) $2\pi(\cos i + i\sin i)$; (d) $-2\pi(\cos i + i\sin i)$; (e) $2\pi(\sin i - i\cos i)$.

(7) Let the Fourier transform of the function $f(t)$ be $F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$; then which of the following statements is true?

(a) The inverse Fourier transform is given by $f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega$;

(b) The Fourier transform of the function $f(t - t_o)$ is $e^{i\omega t_o}F(\omega)$;

(c) The Fourier transform of the function $t^2 f(t)$ is $i^2 F(\omega)$;

(d) The Fourier transform of the function $f(at)$ for $a \in \mathbf{R}$, $a > 0$ is $(1/a)F(\omega/a)$;

(e) Let $G(\omega)$ be the Fourier transform of $g(t)$, then the Fourier transform of $f(t)g(t)$ is $\frac{1}{2\pi}F(\omega)G(\omega)$.

(8) If the Fourier transform of the function $f(t) = 1/(a^2 + t^2)$, $a > 0$, is $F(\omega) = (\pi/a)e^{-a|\omega|}$; then what is the Fourier transform of the function $g(t) = t/(a^2+t^2)^2$?

(a) $-\pi e^{-a|\omega|}$; (b) $-i\pi e^{-a|\omega|}$; (c) $\pi e^{-a|\omega|}$; (d) $\frac{i\pi}{2a}\omega e^{-a|\omega|}$; (e) $\frac{-i\pi}{2a}\omega e^{-a|\omega|}$.

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(9) Let $f(x)$ be an integrable function defined in the interval $-L \le x \le L$, and

$a_o + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$ be the Fourier series representation of $f(x)$ over the interval $-L \le x \le L$; then which of the following statements is true?

(a) a_o is the average value of the function $f(x)$ over the interval $-L \le x \le L$;

(b) The Fourier series converges to $f(x)$ at every interior point $-L < x < L$;

(c) At both end points $x = \pm L$, the Fourier series converge to the same value: $\frac{1}{2}[f(-L)+f(L)]$;

(d) The Fourier series $\sum_{n=1}^{\infty}\left\{\frac{n\pi}{L}\left[-a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right)\right]\right\}$ converges to the function $\frac{df(x)}{dx}$ in the interval $-L<x<L$ except at points where the function $f(x)$ is not continuous;

(e) $\frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_o^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$.

(10) What is the Fourier integral representation of the function $\begin{aligned} f(x) &= 1 \quad \text{for } |x| \le 1 \\ &= 0 \quad \text{for } |x| > 1 \end{aligned}$?

(a) $\frac{2}{\pi}\int_0^{\infty} \sin\omega \cos\omega x\, d\omega$; (b) $\frac{2}{\pi}\int_0^{\infty} \cos\omega \cos\omega x\, d\omega$; (c) $\frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega}{\omega} \cos\omega x\, d\omega$;

(d) $\frac{2}{\pi}\int_0^{\infty} \frac{\sin\omega}{\omega} \cos\omega x\, d\omega$; (e) $\frac{1}{\pi}\int_0^{\infty} \frac{\cos\omega}{\omega} \cos\omega x\, d\omega$.

2. (15%) Write down on the answer sheet the correct answer to each of the following questions. (Derivations are not required.)

(1) Which set or sets of vectors can be a basis for the three-dimensional real space R^3?
- (a) {(1,2,-1), (0,3,1)};
- (b) {(2,4,-3), (0,1,1), (0,1,-1)};
- (c) {(1,5,-6),(2,1,8),(3,0,4),(2,1,1)};
- (d) {(1,0,0),(0,1,0),(0,0,1),(1,1,1)};
- (e) {(1,3,-4),(1,4,-3), (2,3,-11)}.

(2) Which one or ones of the followings can be a real linear vector space?
- (a) the set consisting of all polynomials of degree exactly equal to 2;
- (b) the set consisting of all real solutions $y(x)$ of the ordinary differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$;
- (c) the set consisting of all linear combinations of the vectors (1,0) and (−1,1);
- (d) the set consisting of all vectors parallel to the plane $2x+3y-z=1$ in R^3;
- (e) the set consisting of only the zero vector in R^3.

(3) Which one or ones of the followings can be a real linear vector space and have a dimension of three?
- (a) the set consisting of all polynomials of degree exactly equal to 2;
- (b) the set consisting of all real solutions $y(x)$ of the ordinary differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 3y = 1$;
- (c) the set consisting of all real symmetric 2×2 matrices;
- (d) the set consisting of all linear combinations of the vectors (1,2,1), (0,1,1), and (3,−1, −4) in R^3;
- (e) the set consisting of all vectors $(x, x-y, 0, x+y, z)$ in R^5., where x, y, and z are real numbers.

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3. (15%) Consider the initial-valued problem:

$$\frac{dx}{dt} = x - y \qquad \text{and} \qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \\ dy/dt(0) = 1 \end{cases}$$
$$\frac{d^2y}{dt^2} = -3x - y + \frac{dy}{dt}$$

where $x=x(t)$ and $y=y(t)$.

(a) Define $x_1(t) \equiv x(t)$, $x_2(t) \equiv y(t)$ and $x_3(t) \equiv dy/dt$. Rewrite the above differential equations into three first-order differential equations in terms of x_1, x_2, and x_3 (i.e., a 3×3 matrix equation $X' = AX$ for $X = (x_1(t), x_2(t), x_3(t))^T$).

(b) Find all the eigenvalues of the associated matrix A.

(c) Find all the linearly independent eigenvectors of the matrix A.

(d) Find the exponential of the matrix At (a fundamental matrix of the matrix equation), i.e. $\exp(At)$.

(e) Write down an expression for the solution vector $X(t)$.

4. (10%) A constant coefficient, homogeneous, linear ordinary differential equation has

$$y = x^2\cos(3x)$$

as a particular solution. Let n be the order of the highest derivative appearing in the equation.

(a) What is the smallest possible value of n?

(b) What is the general solution to the ODE that has this smallest possible value of n?

5. (10%) Find an integrating factor of the differential equation

$$(x^2+1)y' + 3x^2y = 6xe^x$$

6. (10%) Solve the one dimensional heat transfer problem

$$u_{xx} = 9u_t, \quad 0 \le x \le 5$$

with boundary conditions $u(0,t) = 0$, $u(5,t) = 4$, $t > 0$ and initial conditions $u(x,0) = 0$, $0 \le x \le 5$.

(a) Find the long time, i.e. time independent, solution reached as $t \to \infty$.

(b) Find the time-dependent solution $u(x,t)$ that satisfies the given boundary and initial conditions.

7. (10%) Solve $u_{tt} = u_{xx} - \pi^2\sin(\pi x)$ for $0 < x < 1$, $t > 0$,

$u(0,t) = u(1,t) = 0$, for $t > 0$,

$u(x,0) = 2\sin(\pi x) + 8\sin(13\pi x) - 3\sin(31\pi x)$, and $u_t(x,0) = -\sin(8\pi x) + 12\sin(88\pi x)$.

試題隨卷繳回

1. Consider the real linear vector space, V, which consists of all real-coefficient polynomials in t of degree ≤ 2. Answer the following questions.
   (1) (3%) What is the dimension of V?
   (2) (3%) Find the components of the vector $k(t) = 1 - 2t + t^2$ with respect to the f-basis $\{f_1(t) = t^2,\ f_2(t) = 2 + t,\ f_3(t) = t - 2t^2\}$ for V. Denote it as $(k)_f$.
   (3) (4%) Find the transformation matrix (P) from the f-basis to the standard basis $\{e_1(t) = 1,\ e_2(t) = t,\ e_3(t) = t^2\}$ for V, that is, $(k)_e = P(k)_f$, where $(k)_e$ is the coordinates of the vector $k(t)$ with respect to the standard basis.

2. (15%) Consider the initial-value problem:

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{d^2y}{dt^2} = x - y + \dfrac{dy}{dt} \end{cases} \quad \text{with} \quad x(0) = 0,\ y(0) = 1,\ \frac{dy}{dt}(0) = 0.$$

   Solve the problem in use of the method of Laplace transform.

3. (15%) Find the general solution of the following ordinary differential equation

$$x\frac{d^2y}{dx^2} + \left(2x^2 - 3\right)\frac{dy}{dx} + \left(x^3 - 2x + 3x^{-1}\right)y = x^6 \quad \text{for } x>0$$

   by performing the change of variables $y(x) = xU(t)$ and $x = \sqrt{t}$.

4. For each of the following Fourier series expansion:

$$f_i(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx \quad \text{for } -\pi < x < \pi,$$

$$f_{ii}(x) = 1 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\left[(2n-1)\frac{x}{2}\right] \quad \text{for } 0 < x < 2\pi;$$

$$f_{iii}(x) = x = k - \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left[(2n-1)\frac{x}{2}\right] \quad \text{for } 0 \le x \le 2\pi$$

   (1) (3%) What is the numerical value of $\int_0^{2\pi} f_{iii}(x)\cos(65x/2)dx$?
   (2) (3%) What is the numerical value of k in $f_{iii}(x)$?
   (3) (3%) What are the numerical values of each series at $x = \pi/3$, π, and 12.5π? (9 answers required)
   (4) (3%) Find the Fourier series for $|x|$, $-2\pi < x < 2\pi$.
   (5) (3%) Does $\int x dx = \dfrac{x^2}{2} = 2\sum_{n=1}^{\infty}(-1)^n n^2(\cos nx - 1)$; $-\pi < x < \pi$?

   Does $\dfrac{dx}{dx} = 1 = 2\sum_{n=1}^{\infty}\cos nx$; $-\pi < x < \pi$?

   Give reasons why you answered "yes" or "no" to these questions.

5. Consider the following 1-dimensional heat equation:

$$\frac{\partial^2 u}{\partial x^2} = u + \frac{\partial u}{\partial t} \quad \text{for } 0 < x < 2,\ t > 0$$

   with initial condition $u(x,0) = \sin\dfrac{\pi x}{4}$, $0 \le x \le 2$.

   (1) (4%) For $t \ge 0$, the wire temperature is kept zero at $x = 0$ and the wire is insulated at $x = 2$. Write down the mathematical form of these boundary conditions.

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(2) (4%) Following (1), use separation of variables on this heat equation to obtain two ordinary differential equations for $X(x)$ and $T(t)$, and the boundary conditions for $X(x)$.
(3) (4%) Following (2), find all non-trivial solutions of $X(x)$.
(4) (3%) Which, if any, of the equations given below is a solution to the given heat equation with the given boundary conditions in part (1). (Justification of your answer is required to get credit.)

(i) $u(x,t)=e^{(-1-\frac{\pi^2}{16})t}\sin\frac{\pi x}{4}$ (ii) $u(x,t)=e^{\frac{-\pi^2 t}{16}}\sin\frac{\pi x}{4}$

(iii) $u(x,t)=e^{\frac{-\pi^2 t}{16}}\cos\pi x$

(iv) $u(x,t)=\sum_{n=1}^{\infty}b_n e^{\frac{-n^2\pi^2 t}{16}}\sin\frac{n\pi x}{4}$, $b_n=\int_0^2\sin(\frac{\pi x}{4})\sin(\frac{n\pi x}{2})dx$

6. Write down the answers to the following questions. (Derivations are not required.)

(1) (3%) Evaluate the line integral $\oint_C \underline{\mathbf{F}}\cdot\underline{\mathbf{n}}\,d\ell$ of a 2-D vector function $\underline{\mathbf{F}}=(\frac{x}{x^2+y^2})\,\underline{\mathbf{i}}+(\frac{y}{x^2+y^2})\,\underline{\mathbf{j}}$ over a closed path C defined by an ellipse $9x^2+4y^2=1$. ($\underline{\mathbf{n}}$ denotes the unit normal vector pointing outwardly along the ellipse.)

(2) (3%) Let $\phi(x,y,z)=xyz$ be a scalar function. Evaluate the surface integral $\oint\!\!\oint_S(\nabla\phi)\cdot\underline{\mathbf{n}}\,dS$ over the bounding surface S of a cube defined by $-1\le x\le 1$, $-1\le y\le 1$, $-1\le z\le 1$. ($\underline{\mathbf{n}}$ denotes the unit normal vector pointing outwardly along the surface of the cube.)

(3) (3%) Let $\underline{\mathbf{F}}=(2x^2-y)\,\underline{\mathbf{i}}+(\cos y-ye^{-y}+4x)\,\underline{\mathbf{j}}$ be a 2-D vector function. Evaluate the line integral $\oint_C\underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ along a closed path C defined by a unit circle centered at the origin.

(4) (3%) Let $\phi(x,y,z)=x^2y-xe^z$. Find the rate of change of $\phi(x,y,z)$ at point (1, 0, -1) along the direction $\underline{\mathbf{u}}=\underline{\mathbf{i}}-2\underline{\mathbf{j}}+\underline{\mathbf{k}}$.

(5) (3%) Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be two continuous and differentiable scalar functions, then $\nabla\cdot(\nabla\phi\times\nabla\psi)=0$. True or False ?

7. Let $z=x+iy$ denote the complex variable, $\bar{z}=x-iy$ be the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. (Derivations are not required.)

(1) (3%) $f(z)=\bar{z}/z$ is an analytic function on the whole z-plane excluding the origin. True or False ?

(2) (3%) Find the residue of the complex function $f(z)=z(z+i)e^{1/z^2}$ at $z=0$.

(3) (3%) Let the Laurent series expansion of $f(z)=(z+3i)/[z(z^2+9)]$ about $z=3i$ be denoted by $\sum_{n=-\infty}^{n=+\infty}c_n(z-3i)^n$ which is a convergent series within the annulus $0<|z-3i|<3$. Find the sum of the coefficients c_n of all negative-power terms; i.e., evaluate $\sum_{n=-\infty}^{n=-1}c_n=?$

(4) (3%) Evaluate the complex integral $\oint_C[(\sin z)/(z-i)^2]dz$ over C: $|z-i|=2$.

(5) (3%) Evaluate the real integral $\int_0^{2\pi}e^{(\cos\theta)}\cos(\sin\theta)\,d\theta$.

<hint>: Consider first the complex integral $\oint_C(e^z/z)dz$ along a unit circle C centered at origin.

試題隨卷繳回

※ 注意：請於試卷上依序作答，並應註明作答之大題及其題號。

1. (10%) The initial conditions $y(0)=y_0$, $y'(0)=y_1$, apply to the following differential equation:
$x^2y''-4xy'+4y=0$.
For what values of y_0 and y_1 does the initial-value problem have a solution ?

2. Consider the matrix $\mathbf{A}=\begin{bmatrix}0&2&1&0\\0&0&2&1\\0&0&0&2\\0&0&0&0\end{bmatrix}$.

(1) (6%) Determine the rank of **A**.

(2) (6%) Let **B** be a 4×3 matrix satisfying $\mathbf{AB}=\mathbf{0}$. Find the maximum possible value of the rank of **B**.

(3) (6%) Find the eigenvalues and eigenvectors of **A**.

(4) (6%) Is **A** diagonalizable ?

(5) (6%) Let $\mathbf{b}=[b_1,b_2,b_3,b_4]^T$. Under what conditions on **b** (if any) does $\mathbf{Ax}=\mathbf{b}$ have a solution ?

3. Let $u(x,t)$ denote the displacement of a finite string over $0<x<\pi$ with fixed ends, $u(0,t)=u(\pi,t)=0$. The string starts to vibrate from its initial states, $u(x,0)=0$ and $u_t(x,0)=(\partial u/\partial t)_{t=0}=x$, after an external force, $F(x)=x(x-\pi)$, is applied onto it. The subsequent string displacement can be described by a 1-D inhomogeneous wave equation:
$$\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}+F(x)\quad \text{, for } 0<t<\infty .$$

(1) (4%) Write $u(x,t)=X(x)T(t)$. Use the homogeneous problem (set $F(x)=0$ with the same boundary conditions) and the Sturm-Liouville theorem to determine the eigenfunction and the corresponding eigenvalue, $\phi_n(x)$ and λ_n, of the problem.

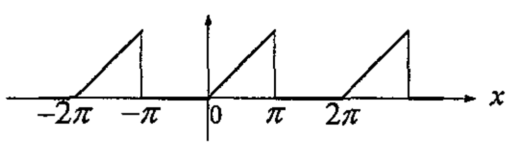
(2) (4%) Find the eigenfunction expansion of the external force: determine the coefficients f_n's in $F(x)=\sum_{n=1}^{\infty} f_n\phi_n(x)$.

Use (1) and (2) to find a solution for the original wave equation in the form of $u(x,t)=\sum_{n=1}^{\infty}T_n(t)\phi_n(x)$:

(3) (4%) Determine a 2[nd] order ODE (ordinary differential equation) that governs $T_n(t)$.

(4) (5%) Apply the initial conditions to solve the ODE obtained in (3). Then, complete the solution to the 1-D inhomogeneous wave equation.

4. A periodic function $f(x)$ is sketched below.



(1) (2%) Write a mathematical description for the function.

(2) (3%) Determine the Fourier series representation, $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(\lambda_n x)+b_n\sin(\lambda_n x)$, of the function: explicitly write out a_0, a_n, b_n, λ_n.

(3) (3%) Determine the fundamental period, ω_0, of the Fourier series. Sketch the amplitude spectrum of the Fourier coefficients over the frequency range of $[0, 4\omega_0]$.

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(4) (2%) What is Gibbs phenomenon ?

(5) (3%) Is Gibbs phenomenon present in the Fourier series representation for $f(x)$ found in (2)? If yes, indicate the location where Gibbs phenomenon will be most pronounced over the range $-5<x<5$. Otherwise, explain why the current Fourier series representation is free of Gibbs phenomenon.

5. Write down the answers to the following questions. (Derivations are not required.)

(1) (3%) Evaluate the distance from the point (1, 3, 0) to the plane: $x-3y+\sqrt{6}z=3$.

(2) (3%) Let $\phi(x,y,z)=xyz+x^2-2y^2$ be a scalar function. Evaluate the flux of $\nabla\phi$ out of the surface of the sphere: $x^2+y^2+z^2=4$.

(3) (3%) Let $\underline{\mathbf{F}}=r\,\underline{\mathbf{e}}_r+r\cos\theta\,\underline{\mathbf{e}}_\theta+z\,\underline{\mathbf{e}}_z$ be a vector function written in the cylindrical coordinates (r,θ,z). Evaluate $\nabla\cdot\underline{\mathbf{F}}$.

(4) (3%) Find the streamlines of the 2-D vector field $\underline{\mathbf{F}}=\sin(2y)\,\underline{\mathbf{i}}+\cos(x)\,\underline{\mathbf{j}}$.

(5) (3%) Let $\phi(x,y)$ and $\psi(x,y)$ be two continuous and differentiable scalar functions on a simple closed curve C and throughout the interior D of C, then $\oint_C -\phi(\partial\psi/\partial y)\,dx+\phi(\partial\psi/\partial x)\,dy=\iint_D \phi\nabla^2\psi\,dA+\iint_D[\quad]\,dA$. Fill in the blank bracket [ ] in the above identity with proper expression.

6. Let $z=x+iy$ denote the complex variable, $\bar{z}=x-iy$ the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. (Derivations are not required.)

(1) (3%) Find the real part of $(1-i)^{(1+i)}$ if the argument θ is restricted in $0\le\theta<2\pi$.

(2) (3%) Find the residue of the complex function $f(z)=z(z-i)\cos(1/z^2)$ at $z=0$.

(3) (3%) Let the Laurent series expansion of $f(z)=(z+i)/(z^2+4)$ about $z=-2i$ be denoted by $\sum_{n=-\infty}^{n=+\infty}c_n(z+2i)^n$ which is a convergent series within the annulus $0<|z+2i|<4$. Find the sum of the coefficients c_n of all negative-power terms; i.e., evaluate $\sum_{n=-\infty}^{n=-1}c_n=?$

(4) (3%) Evaluate the complex integral $\oint_C[\bar{z}/(z+2i)^2]\,dz$ over C: $|z|=1$.

(5) (3%) Evaluate the real integral $\int_{\theta=0}^{\theta=2\pi}\frac{d\theta}{\cos\theta+\sin\theta}$.

台灣大學

應用力學研究所

91~97 學年度

工程數學考古題

1. (30%). Given a forced-vibration system described by

$$\begin{cases} \ddot{x}_1 + 2x_1 - x_2 = a\sin\Omega t \\ \ddot{x}_2 + 2x_2 - x_1 = b\sin\Omega t \end{cases} \tag{1}$$

(a). If we assign $a = 0$, $b = 1$, and $\Omega = 2$ in equations (1), and the initial conditions are $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$, find the solutions of equations (1).

(b). If the forcing frequency Ω is assigned as $\Omega = \sqrt{3}$, What is the relation between a and b in order that the solutions of equations (1) with zero initial conditions are bounded for all times？

2. (35%). Let **A** be an $n \times n$ real matrix. Suppose A is skew-symmetric; i.e. $\mathbf{A}^T = -\mathbf{A}$.

(a). Show that the diagonal elements of **A** are all zero.

(b). Show that $\det \mathbf{A} = 0$ if n is odd.

(c). Show that if **A** has an eigenvalue, the eigenvalue must be zero.
(You can get partial credits by assuming n=3.)

(d). Show that **I+A** is nonsingular (invertible) where **I** is the $n \times n$ identity matrix.
(You can get partial credits by assuming n=3.)

3. (35%). Consider the steady-state temperature distribution of a half-circle region. The region is heated through some boundary conditions. To know the temperature distribution, one needs to consider the following boundary valued problem

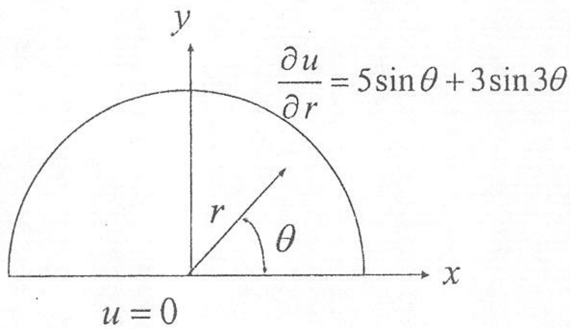
$$\begin{aligned} &\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && 0 < r < 1,\ 0 < \theta < \pi \\ &u(r,0) = u(r,\pi) = 0, && 0 < r < 1 \\ &\frac{\partial u}{\partial r}(1,\theta) = 5\sin\theta + 3\sin 3\theta, && 0 < \theta < \pi. \end{aligned}$$

(a). Find the solution of the above problem.

(b). Evaluate also the following integral

$$\int_C \frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy$$

where C is the boundary of the half-circle region defined above.



試題隨卷繳回

1. (30 %). Let M be a positive-definite symmetric $(n \times n)$ real matrix; i.e., $\mathbf{x}^T \mathrm{M} \mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$ where $\mathbf{x}$ is the $n \times 1$ column vector with real entries.

   (a). (10%). Show that all the eigenvalues of M are positive.

   (b). (10%). Consider the following iterative process: with an initially given unit vector $\mathbf{v}_1$, for $i = 1, 2, 3, \ldots$,
   (i) Compute $\mathbf{u}_i = \mathrm{M}\mathbf{v}_i$,
   (ii) Find $\mathbf{v}_{i+1} = \mathbf{u}_i / \|\mathbf{u}_i\|$,
   where $\|\mathbf{u}_i\|$ denotes the magnitude of $\mathbf{u}_i$. Show that if the above process converges, the sequence $\{\mathbf{v}_i,\ i = 1, 2, 3, \ldots\}$ converges to an eigenvector of M with the associated eigenvalue approached by $\{\|\mathbf{u}_i\|,\ i = 1, 2, 3, \ldots\}$.
   (Note that you may assume $n = 2$ for partial credits.)

   (c). (10%). Solve the following problem

$$\max_{\mathbf{x} \in \mathbf{R}^n} \mathbf{x}^T \mathrm{M} \mathbf{x}, \qquad \text{subject to the constraint} \quad \|\mathbf{x}\| = 2,$$

   in terms of the eigenvector and eigenvalue of M. (Note that you may assume $n = 2$ for partial credits.)

2. (35%).

   (a). Let a be a positive constant and $f(t)$ be a continuous function on $[0,\ L]$, $L > 0$.

   (i). (5%). Show that the only solution of

$$ty' + ay = 0$$

   which is bounded as $t \to 0^+$ is the trivial solution.

   (ii) (10%). Let $f(0) = b$. Show that the equation

$$ty' + ay = f(t)$$

   has a unique solution which is bounded as $t \to 0^+$ and find the limit of this solution as $t \to 0^+$.

   (b). (20%). Given that the equation

$$ty'' - (2t+1)y' + 2y = 0 \qquad (t > 0)$$

   has a solution of the form e^{ct} for some c, find the **general solution**.

3. (35%). Consider the Poisson equation

$$\nabla^2 \varphi(x,y,z) = f(x,y,z)$$

in a bounded domain Ω enclosed by the surface S with unit outward normal $\hat{\mathrm{n}}$.

(a). (15%). If the boundary condition is of the Neumann type; i.e., $\frac{\partial \varphi}{\partial n} = g(x,y,z)$ on the surface S, show that

$$\int_{\Omega} f(x,y,z)dV = \int_{S} g(x,y,z)ds,$$

where dV and ds are the infinitesimal volume and surface elements, respectively. Also provide the physical interpretation of the above equation.

(b). (20%). For simplicity, consider a two-dimensional case in a unit square domain $\Omega = (0,1) \times (0,1)$ as shown in Figure 1. The source term $f(x,y) = 1$, $\forall\, (x,y) \in \Omega$, and the boundary conditions are of the Dirichlet type: $\varphi = -1$ on $x = 0, x = 1$ and $y = 1$; but $\varphi = 0$ on $y = 0$. Solve for $\varphi(x,y)$ and evaluate the normal derivatives of $\varphi(x,y)$ along the boundaries.

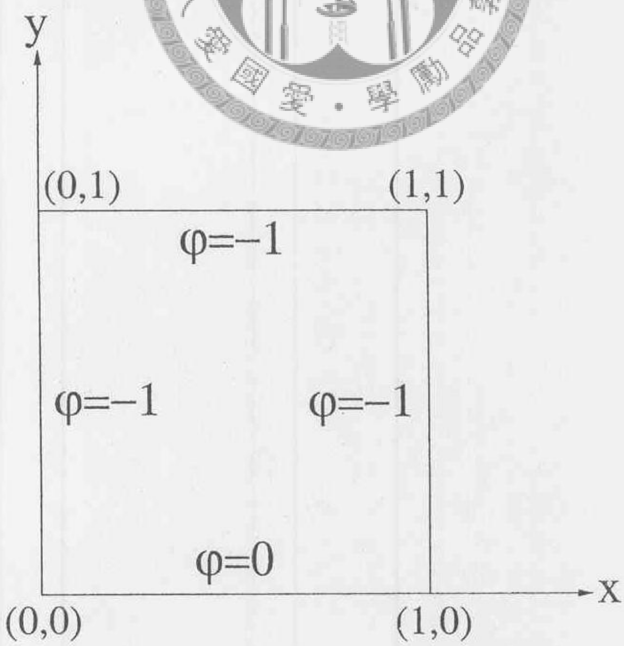


Figure 1: Problem 3(b).

試題隨卷繳回

1. (35%). Suppose $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are square real matrices.

   (a). (10%). Prove that $(\mathbf{AD})^{-1} = \mathbf{D}^{-1}\mathbf{A}^{-1}$ for nonsingular matrices $\mathbf{A}$ and $\mathbf{D}$.

   (b). (15%). For

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & -2 \\ -5 & 3 & 8 \end{pmatrix},$$

   determine the eigenvalues and the corresponding eigenvectors. Also prove that $\mathbf{C}^{-1}\mathbf{AC} = \mathbf{B}$, where

$$\mathbf{C} = \begin{pmatrix} \nu_{1x} & \nu_{2x} & \nu_{3x} \\ \nu_{1y} & \nu_{2y} & \nu_{3y} \\ \nu_{1z} & \nu_{2z} & \nu_{3z} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

   with

$$\begin{pmatrix} \nu_{1x} \\ \nu_{1y} \\ \nu_{1z} \end{pmatrix}, \quad \begin{pmatrix} \nu_{2x} \\ \nu_{2y} \\ \nu_{2z} \end{pmatrix}, \quad \begin{pmatrix} \nu_{3x} \\ \nu_{3y} \\ \nu_{3z} \end{pmatrix}$$

   the eigenvectors corresponding to the eigenvalues λ_1, λ_2 and λ_3, respectively.

   (c). (10%). A surface in space can be expressed as $z = f(x, y)$. Show that

$$-\frac{\partial f}{\partial x}\mathbf{e}_x - \frac{\partial f}{\partial y}\mathbf{e}_y + \mathbf{e}_z$$

   is the normal vector of the surface, with $\mathbf{e}_x, \mathbf{e}_y$ and $\mathbf{e}_z$ the unit vectors along the x, y, and z directions.

2. (30%). Find the solution, if there exists one, of the following differential equations.

   (a). (6%). $\frac{dy}{dx} = y' = \frac{(3x^2+4x+2)}{2(y-1)}$, $\quad y(0) = -1$.

   (b). (8%). $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$.

   (c). (8%). $y'' + 4y = 0, \quad y(0) = 1, \quad y(\pi) = 0$.

   (d). (8%). $y'' + 4y = 0, \quad y(0) = 1, \quad y(\pi) = 1$.

3. (35%). Let S be the boundary surface of an open bounded domain $\Omega \subset I\!R^3$.

   (a). (10%). Let $\mathbf{r} = (x, y, z)$ be the position vector in $I\!R^3$. Show that for $\mathbf{r} \neq (0,0,0)$

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = 0,$$

   where $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

   (b). (10%). Suppose the origin $(0,0,0)$ is contained in the domain Ω. Show that

$$\int_{\Omega} \nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) dxdydz = 4\pi.$$

   (c). (15%). Consider the function

$$f(\mathbf{r}) = \begin{cases} \rho(\mathbf{r}) & \text{if } \mathbf{r} \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

   where $\rho(\mathbf{r})$ is a smooth function. Show that the solution of the Poisson equation

$$\nabla^2 \phi(\mathbf{r}) = f(\mathbf{r}) \qquad \forall\, \mathbf{r} \in I\!R^3$$

   can be expressed as

$$\phi(\mathbf{r}) = \frac{-1}{4\pi} \int_{\Omega} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx'dy'dz',$$

   where $\mathbf{r}' = (x', y', z')$.

試題隨卷繳回

1. (a) (10%) Solve the following initial value problem of $y(x)$ in terms of the given function $f(x)$:

$$y''(x)+4y(x)=f(x), \quad y(0)=y'(0)=0.$$

(b) (10%) Given an initial value problem of $y(x)$ with 2^{nd} order ODE :

$$y''(x)=-\sin y, \quad y(0)=\pi/6, \quad y'(0)=0.$$

Find an approximate (series) solution of $y(x)$ around $x=0$ by Taylor's expansion, up to 3 nonzero terms.

2. (a) (10%) Evaluate the following line integral along the path $(x,y)=(0,0)\rightarrow(2,0)\rightarrow(2,2)$

$$\int_{(0,0)}^{(2,2)}\left(10x^4-2xy^3\right)dx-3x^2y^2dy$$

(b) (10%) Let $f(x)=x^3-2x^2+x-2$. Evaluate $f(\mathrm{A})$ if $\mathrm{A}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$.

3. (30%). Consider the following initial-boundary value problem

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)=\dfrac{\partial^2 u}{\partial x^2}(x,t)+\delta(x-b), \quad 0<x<L,\ 0<b<L,\ t>0\\ u(0,t)=0,\ t>0\\ u(L,t)=0,\ t>0\\ u(x,0)=0,\ 0<x<L\end{cases} \qquad \text{(A)}$$

where $\delta(x)$ is the Dirac delta function. Define $\bar{u}_n(t)=\int_0^L u(x,t)X_n(x)dx$, $n=1,2,3,\cdots$, where $X_n(x)$ is defined by

$$X_n(x)=\sqrt{\frac{2}{L}}\sin\lambda_n x, \quad \lambda_n=\frac{n\pi}{L}$$

(a) (5%). Let $f(x)$ be a piecewise smooth function defined on $[0,L]$. Write down the standard formula of the Fourier sine series of $f(x)$ and cast it into the form as

$$f(x)=\sum_{n=1}^{\infty}\bar{f}_n X_n(x), \quad \bar{f}_n=\int_0^L f(x)X_n(x)dx$$

(b) (10%). Multiply by $X_n(x)$ on both sides of equation (A) and integrate with respect to x from 0 to L; i.e.,

$$\int_0^L\frac{\partial u}{\partial t}(x,t)X_n(x)dx=\int_0^L\frac{\partial^2 u}{\partial x^2}(x,t)X_n(x)dx+\int_0^L\delta(x-b)X_n(x)dx$$

Show that the problem reduces to

$$\begin{cases}\dfrac{d\bar{u}_n}{dt}(t)+\lambda_n^2\bar{u}_n(t)=X_n(b)\\ \bar{u}_n(0)=0,\end{cases}$$

(c) (10%). Find the solution $\bar{u}_n(t)$ of (b).

(d) (5%). Using the results of (a) and (c), find the solution of the original initial-boundary value problem of (A).

4. (30%) Let Fourier transform pair in x with transform parameter k define as

$$\begin{cases}f(x)\equiv\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\bar{f}(k)e^{-ikx}dk\\ \bar{f}(k)\equiv\displaystyle\int_{-\infty}^{\infty}f(x)e^{ikx}dx\end{cases}$$

Let Fourier transforms of $f(x)$ and $g(x)$ exist, and be denoted as $\bar{f}(k)$ and $\bar{g}(k)$, respectively.

(a) Show that the Fourier transform of $f(ax)$ can be expressed as $\bar{f}(k/a)/a$, where constant $a>0$.

(b) Show that the Fourier transform of the first derive of $f(x)$, i.e. $f'(x)$, can be expressed as $-ik\bar{f}(k)$.

(c) If $f(x)$ is a real function in x, show that $\bar{f}(-k)=\bar{f}^*(k)$, where * denote as complex conjugate.

(d) If $f(x)$ is an even function in x, show that $\bar{f}(k)=\bar{f}^*(k)$.

(e) Show that the Fourier transform of $\int_{-\infty}^{\infty}f(\xi)g(x-\xi)d\xi$ can be expressed as $\bar{f}(k)\bar{g}(k)$

(f) Show that $\int_{-\infty}^{\infty}|f(x)|^2dx=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left|\bar{f}(k)\right|^2dk$, where $|\ |^2$ stands for the squared norm of a complex function, e.g. $|f(x)|^2\equiv f(x)f^*(x)$

試題隨卷繳回

1. (35%) Consider a mass m_1 suspended vertically from a rigid support by a weightless spring with a second mass m_2 suspended from the first mass by means of a second weightless spring as shown in the figure 1. Let's treat the two masses as point masses, and assume that the two springs obey Hooke's law with spring constants k_1 and k_2, respectively.

(i) Derive the equations governing the displacements for the two masses, $x_1(t)$ and $x_2(t)$, using Newton's second law with negligible air resistance. Here t denotes the time. The result should be

$$m_1 \frac{d^2x_1}{dt^2} = -k_1 x_1(t) + k_2[x_2(t) - x_1(t)],$$
$$m_2 \frac{d^2x_2}{dt^2} = -k_2[x_2(t) - x_1(t)].$$
(10%)

(ii) Solve the equations for $x_1(t)$ and $x_2(t)$ in (i) with $m_1 = m_2 = 1,\ k_1 = 5,\ k_2 = 6$, subject to $x_1 = 2,\ \frac{dx_1}{dt} = 5,\ x_2 = -10,\ \frac{dx_2}{dt} = 1$ at $t = 0$. Calculate the speeds of the two masses. (15%)

(iii) How do we modify the equations in (i) if an external force F(t) is applied to mass m_2 along the direction of x_2 and the drag associated with the air resistance is included? Assume that the drag on the mass is proportional to its velocity but in opposite direction. (10%).

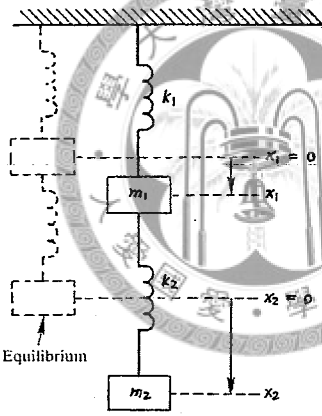


Figure 1

2. (30%) Consider a full-rank ($m \times n$) real-valued matrix $\mathbf{A}$ ($m \ge n$). Define $\text{Range}(\mathbf{A}) = \{\mathbf{y} \in R^m : \exists \mathbf{x} \in R^n \ni \mathbf{y} = \mathbf{A}\mathbf{x}\}$ and $\text{Ker}(\mathbf{A}^T) = \{\mathbf{y} \in R^m : \mathbf{A}^T\mathbf{y} = \mathbf{0}\}$, where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$, and R^m is the m-dimensional vector space of real numbers. Show that

(i) Any vector in R^m can be expressed as a sum of a vector in $\text{Range}(\mathbf{A})$ and a vector in $\text{Ker}(\mathbf{A})$. (10%)

(ii) $\text{Range}(\mathbf{A}) \cap \text{Ker}(\mathbf{A}) = \{\mathbf{0}\}$. (10%)

(iii) Either the solution of $\mathbf{A}\mathbf{x} = \mathbf{y}$ exists for all $\mathbf{y} \in R^m$, or the problem $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has non-zero solutions. (10%)

3. (35%)

(i) (a) Determine the regions where the following partial differential equation is of elliptic, parabolic, or hyperbolic type. (6%)

$$u_{xx} + yu_{yy} = 0$$

(b) Obtain its characteristics and its canonical form for each region in (a). (9%)

(ii) Solve the partial differential equation

$$u_{tt} - u_{xx} = 0,\ (0 < x < \infty),$$

subject to the conditions

$$u(0,t) = H(t)\exp(-t),\ u(x,0) = u_t(x,0) = 0,$$

where $H(t)$ is the Heaviside unit step function. (20%)

1. (24 %) $A=\begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix}$,

(a) (6%) find the eigenvalues and eigenvectors of A and A^{-1}, respectively.

(b) (6 %) calculate A^7 and A^{-7}.

(c) (6%) find the eigenvalues and eigenvectors of A^7 and A^{-7}, respectively.

(d) (6 %) for any vector $x=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, show that $x^T A^7 x$ and $x^T A^{-7} x$ are always positive unless x is a zero vector.

2. (6%) Evaluate the integral $\int_{(0,1)}^{(1,0)}(4x^3-3x^2y^2)dx-2x^3y\,dy$ along the path $x^2+y^2=1$.

3. (40%). Let $\tilde{f}(s)=\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt$ be the Laplace transform of $f(t)$.

(a). (5%). Find out the Laplace transform of the function $\frac{1}{2}t^2$.

(b). (10%). Show that

$$\mathcal{L}[f(t-a)H(t-a)]=e^{-as}\tilde{f}(s),$$

where $a>0$ is a constant and $H(t)$ is the Heaviside step function.

(c). (10%). Consider the following initial boundary value problem

$$\begin{aligned}&\tfrac{\partial^2 u}{\partial t^2}(x,t)=\tfrac{\partial^2 u}{\partial x^2}(x,t)+1, \quad 0<x<\infty, \quad t>0,\\ &u(0,t)=0, \quad t>0,\\ &|u(x,t)| \text{ is bounded as } x\to\infty, \quad t>0,\\ &u(x,0)=0, \quad x>0,\\ &\tfrac{\partial u}{\partial t}(x,0)=0, \quad x>0.\end{aligned}$$

Define

$$\tilde{u}(x;s)=\int_0^\infty u(x,t)\,e^{-st}dt.$$

Using the technique of Laplace transform to show that

$$\begin{aligned}&\frac{d^2\tilde{u}}{dx^2}(x;s)-s^2\tilde{u}(x;s)+\frac{1}{s}=0,\\ &\tilde{u}(0;s)=0, \quad |\tilde{u}(x;s)| \text{ is bounded as } x\to\infty.\end{aligned}$$

Note that here the variable s is treated as a constant.

(d). (10%). Find the solution $\tilde{u}(x;s)$ in (c).

(e). (5%). Using (a) and (b) to find the solution $u(x,t)$ of the initial boundary value problem (c).

見背面

4. **(30%)** An undamped spring-mass system , of which m_1 is mass of the main body, k_1 the spring constant, and $f(t)$ the applied excitation, is shown in Fig. 4a, this is referred to as the primary system. The equation of vibration by Newton's second law of motion is an ordinary differential equation as

$$\ddot{x}+\omega^2 x = f(t)/m_1 . \qquad (4\text{-}1)$$

where $\omega=\sqrt{k_1/m_1}$.

(a) (12%) Show that by using *the method of variation of parameters* the general solution of equation (4-1) including both the homogeneous and particular solutions is of the form

$$x = A\sin\omega t + B\cos\omega t + \frac{1}{\omega m_1}\int_0^t f(\tau)\sin\omega(t-\tau)\,d\tau , \qquad (4\text{-}2)$$

where A and B are integration constants determined by initial conditions.

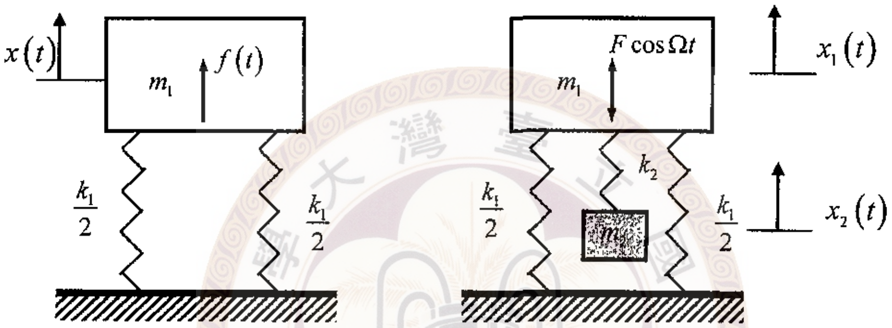


Fig. 4a **Fig. 4b**

(b) (6%) If $f(t)$ is a harmonic excitation given by $f(t)=F\cos\Omega t$ where Ω is the excitation frequency, show that the general solution is in the form

$$x(t) = A\sin\omega t + E\cos\omega t - \frac{F}{m_1(\Omega^2-\omega^2)}\cos\Omega t , \qquad (4\text{-}3)$$

where A and E are constant.
The third term on the right of equation (4-3) is also called the *steady-state solution*.

(c) (7%) One method of reducing the vibration amplitude of the primary system subjected to harmonic excitation is to attach a tuned vibration absorber, which is a second spring-mass system, as shown in Fig. 4b. The equations of motion of the two degree-of-freedom system, written in matrix form, are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}\cos\Omega t . \qquad (4\text{-}4)$$

Let the *steady-state solution* be given by

$$x_1 = U_1\cos\Omega t , \qquad x_2 = U_2\cos\Omega t \qquad (4\text{-}5)$$

and show that

$$U_1 = \frac{(k_2-m_2\Omega^2)F}{D(\Omega)}, \qquad U_2(\Omega)=\frac{k_2 F}{D(\Omega)}, \qquad (4\text{-}6)$$

where

$$D(\Omega) = (k_1+k_2-m_1\Omega^2)(k_2-m_2\Omega^2)-k_2^2 .$$

(d) (5%) How should we choose the values of m_2 and k_2 for a given value of Ω so that the amplitude U_1 is reduce to aero?

試題隨卷繳回

1. (10%). Consider a 2×2 matrix **A** whose eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$. The corresponding eigenvectors are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

what is **A**?

2. Suppose

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

where a_i and b_i are all real numbers. Consider a 3×3 matrix $\mathbf{C} = (c_{ij})$ where $i, j = 1, 2, 3$. Suppose $c_{ij} = a_i b_j$ for $i, j = 1, 2, 3$.

(a).(10%). Find the determinant of **C**.

(b).(10%). Assume $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}^T\mathbf{b} \neq 0$. Find the real eigenvalues and eigenvectors of **C**.

3. (a). (8%). Given a vector field $\mathbf{u} = (xy-1)\mathbf{i} - xz\mathbf{j} + (2-yz)\mathbf{k}$, find a vector field $\mathbf{w}$ such that $\nabla \times \mathbf{w} = \mathbf{u}$. Is $\mathbf{w}$ unique? Why?

(b).(8%). Evaluate the volume of the solid bounded by the cylinder $r = 2\cos\theta$, the cone $z = r$, $(r \geq 0)$, and the plane z=0.

(c).(8%). Evaluate $\lim_{x\to\infty} \frac{\ln x}{x^\alpha}$ and $\lim_{x\to 0} x^\alpha \ln x$ for any $\alpha > 0$.

(d).(8%). Given an algebraic equation $Ax = b$, where A, x, and b are respectively $n\times n$, $n\times 1$ and $n\times 1$ arrays. Let the sum of total number of algebraic operations including '+', '-', '×' and '/' for solving this equation by Gaussian elimination be N. It is known that $N = a_3 n^3 + a_2 n^2 + a_1 n + a_0$ for all n. Find a_0, a_1, a_2 and a_3 by considering $n \leq 3$.

4. (18%). Determine the response of the damped vibrating system corresponding to the equation

$$y'' + 3y' + 2y = r(t),$$

where $r(t) = 1$ when $0 < t < 1$ and 0 otherwise; assume that $y(0) = 0$ and $y'(0) = 0$. Also give physical interpretation of each term in the above ordinary differential equation.

5. (20%) Consider the heat flow in an infinite bar governed by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,$$

with initial condition $u(x,0) = f(x)$. Solve for $u(x,t)$. Give physical interpretation of the problem and your solution.

試題隨卷繳回