元智大學

通訊工程學系碩士班

91~97 學年度
工程數學考古題

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pe91014

1. Suppose phone calls arriving at a switchboard have a Poisson distribution with parameter λ per minute, and the probability of no calls during one minute is 1/3.
(a) Please find the expected waiting time between the arrivals of two calls. (10/100)
(b) Please derive the probability that at least 2 calls arrive at the switchboard during 1 minute period. (10/100)

2. Let the pdf of X be $f_X(x) = \frac{1}{\sqrt{\pi}}\exp(-x^2)$ for $-\infty \leq X \leq \infty$. Please find the pdf of $Y = \ln X$. (10/100)

3. The random variables X and Y have a joint probability density function (pdf) given by

$$f(x,y) = \begin{cases} 20A - Ax - 2Ay, & \text{if } 0 \leq x \leq 5;\ 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

where A is a constant.
(a) Please find the value of A. (10/100)
(b) Please derive the conditional pdf $f(Y|X)$. (10/100)
(c) Please find the correlation between X and Y. (10/100)

4. Consider the vector space of polynomials P_2, where each vector is a second-order polynomial $a_0 + a_1t + a_2t^2$, $a_i \in R$, $i = 0,1,2$.
(a) Please find a basis for P_2. What is the dimension of the vector space? (8/100)
(b) Consider the set of polynomials $(t-1)^2, 3+t-2t^2, -3+3t+2t^2$. Find the coordinate vectors for the polynomials with respect to the basis in a).
Is the set of polynomials linearly independent? Why? (12/100)

5. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

(a) Consider the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Is it in A's null space? Is it in A's column space? Find the inverse of A, if it exist. (10/100)

(b) The vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ can be decomposed into the sum of two orthogonal vectors, $\vec{v} = \vec{c} + \vec{d}$, where $\vec{c}$ is the orthogonal projection of $\vec{v}$ onto A's second column $\vec{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$. Please find $\vec{c}, \vec{d}$. (10/100)

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（本科試題若題意不清，可自行假設條件。若懷疑題目有錯誤，可修正之，但須寫出。監試人員可不作任何題意說明）

1. Please solve the eigenfunctions as following (20%)

$$\frac{d^4u}{dt^4} = -k^2\frac{d^2u}{dt^2}, \quad 0 \le t \le \omega$$

and $\frac{du}{dt}(0) = 0$, $\frac{d^2u}{dt^2}(\omega) = 0$, $u(0)=0$, $u(\omega) = 0$.

2. Please solve the following equations using Laplace transform. (20%)

$$\frac{d^2z}{dt^2} + 4z = 3\frac{dz}{dt}, \quad \frac{dz}{dt}(0) = 5, \quad z(0)=1.$$

3. Please solve the differential equation. (10%)

$$\frac{d^2u}{dt^2} - t\cos 3t = -9u$$

4. (a)Derive mean and variance of an uniform random variable?
   (your answer cannot be zero)
   (b) Derive mean and variance of a Gaussian random variable?
   (your answer cannot be zero) (10%)

5. When a marix B is symmetric, $B^T=B$.
   Write the formula (or property) to be satisfied when **B** is a
   (a) Skew symmetric matrix
   (b) Orthorgonal matrix
   (c) Hermitian matrix
   (d) Skew (or Anti) Hermitian matrix
   (e) Unitary matrix (10%)

6.(a) Describe the condition to utilize Fourier sine series expansion and the condition to utilize Fourier cosine series expansion.
  (b) What is the complex (exponential) form of Fourier series expansion?. (10%)

7.(a) Explain positive definite
  (b) Explain negative semi-definite. (10%)

8. (a)What is Cauchy-Riemann condition?
   (b)What is Residue theorem? (10%)

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1. The sample space of a random experiment contains three equally likely outcomes a, b, c. Consider the following random variables. X(a)=0, X(b)=0.5, X(c)=0.5. Y(a)=0, Y(b)=0, Y(c)=1. (25%)
   a) Plot the probability mass function and cumulative distribution function for each random variable. Find the mean, variance for each random variable.(15 %)
   b) Find the joint probability mass function for X,Y. Are X, Y independent? Uncorrelated? Find the correlation coefficient of X, Y. (10%)

2. Random variables X, Y are uniformly distributed over the region $0 \le x \le y \le 4$. (25%)
   a) Find and sketch the marginal densities $f_X(x), f_Y(y)$ and the conditional densities $f_{X|Y}(x|y), f_{Y|X}(y|x)$. (10 %)
   b) Suppose W=min(X,Y). Conditional on the event W<3, repeat part a). (15 %)

3. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $w = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

   Consider the linear transformation $x \to Ax$, where $A = [v_1, v_2, v_3]$.
   Is the linear transformation invertible? Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Find a basis for the subspace spanned by $\{v_1, v_2, v_3\}$. Is the basis unique? (25%)

4. Answer true or false for the following questions. (Explanations or counterexamples are required when the answer is 'false'.) (25%)
   a) If the columns of A are linearly independent, then AB must have linearly independent columns for any matrix B. (5%)
   b) If matrix $A_{n\times n}$ is row equivalent to the identity matrix I_n, then all the eigenvalues of $A_{n\times n}$ are 1. (5%)
   c) If the eigenvalues of a matrix $A_{n\times n}$ are not distinct, then the number of linearly independent eigenvectors is less than n, and the eigenvectors cannot form a basis of R^n. (5%)
   d) The eigenvalues of a matrix are on its main diagonal. (5%)
   e) The set of solutions of the equation $x_2=x_1+3$ form a subspace of R^2. (5%)

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(1) (a) Consider a second order differential equation

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

with real coefficients a_0 and a_1. Please show the conditions under which the two linearly independent solutions have forms of e^{ax} and xe^{ax}. Also prove these two solutions. (hint: use reduction of order method) (20%)

(b) Now consider a second order differential equation

$$x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = 0$$

with real coefficients a_0 and a_1. Please show the conditions under which the two linearly independent solutions have forms of x^a and $x^a \ln|x|$. Also prove these two solutions. (hint: use reduction of order method or the result of (a)) (15%)

(2) Solve the initial value problem

$$y'' + 4y' + 3y = e^{-x} + e^{-3x}$$

with $y(0) = 2$ and $y'(0) = 1$. (15%)

(3) (a) (10%)Find the Laplace transform of

$$\frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at$$

(b) (10%)Find the inverse Laplace transform of

$$\frac{2a^2 s}{(s^4 - a^4)} \quad ; \quad s > |a|$$

(4) (20%) Use Laplace transform to following differential equation

$$y'' + 4y' - 21y = 2e^{-2t}\sin 3t$$

with $y(0) = 1$ and $y'(0) = 0$

(5) Find the Fourier series of $x\sin x$, $-\pi < x < \pi$ (10%)

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(請命題教授 務必註明)

(1) (10 points) Use matrix operations to find the solutions of following matrix equation Ax=b:

(a) $A = \begin{bmatrix} 1 & 2 & 1 & 7 & 0 \\ 3 & 6 & 4 & 24 & 3 \\ 1 & 4 & 4 & 12 & 3 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

How many solutions you have found?

(b) $A = \begin{bmatrix} 2 & 3 & 6 & 1 \\ 1 & 4 & 2 & 2 \\ 4 & 11 & 10 & 5 \\ 1 & 0 & 1 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$

How many solutions you have found?

(2) (20 points) Use Laplace Transform to solve the following system of differential equations:

$$x' + x - y = 2$$
$$y' - y + 2z = 0$$
$$z' + x - y = \cos t$$

with given initial conditions of $x(0) = 1, y(0) = 0, z(0) = 2$

(3) (30 points)

(a) The Fourier series representation of $f(x)$ over the interval $-\pi \le x \le \pi$ can be expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where (a_n, b_n) are unknown coefficients to be found. Please show the each pair of basis functions are orthogonal.

(b) Find the Fourier series representation of

$$f(x) = \begin{cases} \cos 2x & -\pi < x < -\pi/2 \\ 0 & -\pi/2 \le x \le 0 \\ \cos 2x & 0 < x \le \pi \end{cases}$$

(4) Consider the following Bessel's equation:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0; \quad x \ge 0$$

where $\nu \ge 0$ is an arbitrary real number.

(a) (5 points) Is $x = 0$ a regular point or regular singular point? Under what condition of ν?

(b) (20 points) Use power series expansion to solve the differential equation (assume the first term, $a_0 \ne 0$, in the power series)

(c) (5 points) What are the initial values at $x = 0$?

(5) (10 points) Find the general solution of

$$y^{(3)} - 5y'' + 6y' = 2x^2 + \cos x$$

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1) A point is uniformly distributed within the unit circle. In other words, its location has the following probability density $f_{X,Y}(x,y)=\begin{cases} C; & 0 \le x^2+y^2 \le 1 \\ 0; & otherwise \end{cases}$ (20/100)

a) Find the probability that its distance from the origin is less than z, $0 \le z \le 1$. (10/100)

b) Find the correlation, covariance, and correlation coefficient of X,Y. (10/100)

2) The probability density function of random variable X is $f_X(x)=0.5\delta(x)+0.5[e^{-(x-2)}]u(x-2)$ where $\delta(x)$ is the unit impulse function, and $u(x)$ is the unit step function. Find the expected value and the CDF of X. Sketch the CDF. (25/100)

3) Consider the transformation between vector space of polynomials. $T: P_2 \to P_1$;
$T(at^2+bt+c)=2at+b$ for all $at^2+bt+c \in P_2$

Is the transformation linear? (If your answer is no, provide a counterexample. If your answer is yes, provide a proof.) Is the transformation one-to-one? Onto? Explain. (25/100)

(credit will be given only when counterexample/proof /explanation are provided)

4) Suppose a signal waveform $f(t)$ has the spectrum $F(w)=1$, $|w|<1$ and $F(w)=-2, 1 \le w \le 2$, or $-2 \le w \le -1$, otherwise $F(w)=0$. (30/100)

a) Find the smallest frequency w_0 such that the interval $[-w_0, w_0]$ contains half of the total energy of $f(t)$. (10/100)

b) Sketch the amplitude spectrum and phase spectrum in the interval [-10,10] when the signal is sampled with sampling frequency $w_s = 2(rad/\sec)$. (10/100)

c) Find the smallest sampling frequency w_s such that the sampled version of the signal can be reconstructed perfectly. Describe how you would reconstruct the signal waveform from the sampled sequence. In real life, can the signal be perfectly reconstructed? Why or why not? (10/100)

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1. There are two identically distributed Bernoulli random variables X, Y with P[X=0]=P[Y=0]=P[X=1]=P[Y=1]=1/2. (25/100)
   (a) Find the probability P[X=1,Y=1]. If the answer is not unique, find the minimum and maximum possible values. (8/100)
   (b) Let X, Y be independent and Z=max(X,Y). Find the joint probability mass function $P_{X,Z}(x,z)$ and the correlation coefficient of X, Z. (9/100)
   (c) For Z in part (b), find and sketch the probability density function $f_Z(z)$ and the cumulative distribution function $F_Z(z)$. (8/100)

2. Matrix $C = B_{3\times 2}A_{2\times 3}$. For each of the descriptions below, answer whether it is possible or not possible. If possible, give an example. If not, explain. (25/100)
   (a) C has linearly independent columns. (8/100)
   (b) C has linearly independent rows. (8/100)
   (c) Linear transformation $\bar{y} = T(\bar{x}) = C\bar{x}$ is invertible. (9/100)

3. Consider the collection of complex sinusoids $\{e^{jnw_0t}, n \in Z\}$, $w_0 = \pi$ (rad/sec). (30/100)
   (a) Find the smallest common period T of these sinusoids. (10/100)
   The collection of sinusoids can be used to synthesize arbitrary periodic signals with period T through their Fourier series expansions. Can they be used to synthesize arbitrary periodic signals with period 2T? (1/2)T? Why?
   (b) Give the periods, frequencies, and average powers of the two sinusoids e^{jw_0t}, e^{-jw_0t}, respectively. Are they orthogonal? Derive your answer. (10/100)
   (c) Find and plot the spectrum X(w) versus w for the signal x(t)=e^{-jw_0t}, where
   $x(t) = (1/2\pi)\int_{-\infty}^{\infty} X(w)e^{jwt}\,dw$. (10/100)

4. A signal $x(t) = e^{(-2+j3)t}u(t)$, $u(t)$ is the unit step function. (20/100)
   (a) Sketch the imaginary part of $x(t)$ versus t, and the magnitude $|x(t)|$ versus t. Compute the total energy of $x(t)$. (10/100)

   (b) Find the spectrum X(w) of $x(t)$. Plot the amplitude spectrum |X(w)|. Is it theoretically possible to reconstruct this signal after sampling it? If so, what is the minimum required sampling frequency? If not, why? (10/100)

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系(所)別： **通訊工程學系** 組別： **微波工程組** 科目： **工程數學** 用紙第 1 頁共 1 頁

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1. Show the homogeneous and particular solutions of the following linear first order differential equation.

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

Please show detailed derivations. (10 points)

2. Given the second order differential equation:

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

(a) Let's assume $y_1(x)$ and $y_2(x)$ be the homogeneous solutions (i.e., solutions of (2) given $f(x) = 0$), please show (10 points)

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx \tag{3}$$

(b) Find the particular solution, $y_p(x)$, of (2) in terms of $y_1(x)$ and $y_2(x)$. (10 points)

3. (a) Show the convolution theorem in Laplace transform (denoted by $L\{.\}$): if $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$ and of exponential order, then

$$L\{f * g\} = L\{f\}L\{g\} = F(s)G(s) \tag{4}$$

where $F(s) = L\{f\}$ and $G(s) = L\{g\}$. (10 points)

(b) Evaluate $L\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}$.(10 points)

4. Compute e^{At} for $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. (Hint: solve a system of differential equations that has a solution of e^{At}) (20 points)

5. (a) Solve the following differential equation: (10 points)

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + k^2\right)E(x,y,z) = 0 \tag{5}$$

(b) Solve (a) and apply boundary conditions: $E(x,y,z) = 0$ at $x = 0, a$ (It is a parallel plate condition) (10 points)

(c) Solve (a) and apply boundary conditions: $E(x,y,z) = 0$ at $x = 0, a$, and $E(x,y,z) = 0$ at $y = 0, b$. (It is a situation of a rectangular waveguide) (10 points)

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(1) Consider an exponential random variable X with E[X]=1/3.
   (a) Find the conditional probabilities P[X>6|X>2], P[X=3|X>2]. (5/100)
   (b) Find the conditional probability density function $f_{X|X<2}(x)$ and the conditional mean E[X|X<2]. (10/100)
   (c) Suppose Y is another random variable that is independent with X and identically distributed as X. W=max(X,Y). Find the cumulative distribution function and the probability density function of W. (10/100)

(2) T: V→W is a transformation from arbitrary vector space V to vector space W.
   (a) Describe the conditions required for T to be a linear transformation.
   Describe the conditions required for T to be invertible. (12/100)
   (b) In the special case when V= R^n, W= R^m, how would (a), (b) translate into matrix/vector properties? Give an example of T that is linear but not invertible, and also give an example of T that is invertible but not linear. (13/100)

(3) (a) For the following matrices, find all eigenvectors, eigenvalues, and describe a basis formed by eigenvectors, whenever possible. (12/100)

$$A=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix},\quad B=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix},\quad C=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}.$$

   (b) What are the meanings and importance of eigenvectors and eigenvalues? What is the advantage of having enough eigenvectors to form a basis? Is it always possible to find enough eigenvectors to form a basis? (13/100)

(4) Answer the questions below for the signal

$$x(t)=\cos(0.2\pi t)-\sin(0.2t+\pi/4)+0.2$$

   (a) Is $x(t)$ periodic? If so, find the period and the corresponding fundamental frequency w_0. If not, explain. (10/100)
   (b) Find the spectrum of $x(t)$, also plot the magnitude spectrum and phase spectrum, respectively. Discuss the differences between spectrums of periodic signals and aperiodic signals. (15/100)

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1. Fourier series expansion (30 points)

(a)、Consider a set of functions $\{1, \cos x, \cos 2x, \cdots, \sin x, \sin 2x, \cdots\}$ with a definition interval $x \in [-\pi, \pi]$, please show that each pair of the functions are orthogonal with respect to the following definition of inner product:

$$< f(x), g(x) >= \int_a^b f(x)g(x)dx \text{ with } x \in [a,b] \text{ for } f(x) \text{ and } g(x). \tag{1}$$

For an arbitrary $h(x)$, $x \in [-\pi, \pi]$, which is represented by

$$h(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx], \tag{2}$$

please find the coefficients a_0, a_i and b_i.

(b)、Next, consider $r(x)$, $x \in [-p, p]$, please extend (a) to define a set of Fourier series functions $\{\phi_1(x), \phi_2(x), \cdots \phi_n(x), \cdots\}$ that can completely represent $r(x)$ in terms of a Fourier series expansion. Find the coefficients.

(c)、Next, consider $q(x)$, $x \in [0, L]$, please extend (a) and (b) to define a set of Fourier series functions $\{\varphi_1(x), \varphi_2(x), \cdots \varphi_n(x), \cdots\}$ that can completely represent $q(x)$ in terms of a Fourier series expansion. Find the coefficients.

2. Fourier Transform (20 points)

Consider a Fourier transform defined by

$$F\{g(t)\} \equiv G(\omega) = \int_{-\infty}^{\infty} g(t) e^{j\omega t} dt \tag{3}$$

(a) (10 points) Please show that

$$F\{g^{(n)}(t)\} = (-j\omega)^n G(\omega) \tag{4}$$

(b) (5 points) Let $g(t) = \sin(at) + at\cos(at)$, please find $G(\omega)$?

(c) (5 points) Let

$$p(t) = \begin{cases} 1 & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{else} \end{cases}. \tag{5}$$

Please find $P(\omega) = F\{p(t)\}$.

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3. System of differential equations (25 points)

Please use eigenvalue and eigenvector decomposition to solve the following matrix equation.

(a) (10 points) Solve $X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$ (6)

(b) (15 points) Show that $X' = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} X$ has an eigenvalue λ_1 of multiplicity 5. How many linearly independent eigenvectors corresponding to λ_1 can you find? What are them?

4. (15 points) Laplace Transform

Use Laplace transform to solve the following system of differential equation:

$$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} = 0 \\ \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 4\dfrac{dx}{dt} = 0 \end{cases} \quad \text{with initial conditions} \quad \begin{cases} x(0) = 1, \; x'(0) = 0 \\ y(0) = -1, \; y'(0) = 5 \end{cases} \qquad (7)$$

5. (10 points)

Solve the following third order differential equation:

$$y''' + 8y = 2x - 5 + 8e^{-2x}; \quad y(0) = -5, y'(0) = 3, y''(0) = -4 \qquad (8)$$

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(1) Consider the vector space of nth order polynomials, P_n where the standard basis is $\{x^n, x^{n-1}, \ldots, x^2, x, 1\}$. Let D be the operation of taking the derivative, D^2 be the operation of taking the second derivative. (26/100)

(a) Consider the transformation D^2+D+2 of P_n into itself. Prove that the transformation is linear. (13/100)

(b) Find the kernal(null space) and range of this transformation. (13/100)

(2) Let matrices $A_{m\times n}, B_{n\times r}, C_{m\times r}$ be the matrices associated with three linear transformations, $C = AB$. Answer the following questions. If your answer is "yes", explain. If your answer is "no", give a counterexample. (24/100)

(a) If the linear transformations associated with A and B are both one-to-one, is it true that C is also one-to-one? (6/100)

(b) If the linear transformations associated with A and B are both onto, is it true that C is also onto? (6/100)

(c) If the linear transformations associated with A is one-to-one, with B is onto, is it true that C must be one-to-one? Onto? (6/100)

(d) If the linear transformations associated with A is onto, with B is one-to-one, is it true that C must be onto? One-to-one? (6/100)

(3) Let X, Y be two independent, discrete random variables with known cumulative distribution functions. (24/100)

(a) Let Z=X+Y. Described the probability mass function of Z in terms of the cumulative distribution functions of X, Y. (6/100)

(b) Let W=max(X, Y). Describe the cumulative distribution function of W in terms of the cumulative distribution functions of X, Y. (6/100)

(c) Let S=min(X,Y). Describe the cumulative distribution function of S in terms of the cumulative distribution functions of X, Y. (6/100)

(d) Prove that $E[S] \le \min(E[X], E[Y])$, $E[W] \ge \max(E[X], E[Y])$. (6/100)

(4) Suppose the time until the next eruption of the famous volcano that destroyed Pompeii, Mount Vesuvius, is modeled as an exponential random variable X with an average of 100 years starting from the last eruption in 1944. (26/100)

(a) Given that Vesuvius has not erupted up to 2006, i.e. X>62, let Y=X-62 be the remaining time until the next eruption, find the average remaining time until the next eruption, E[Y|X>62]. (13/100)

(b) Suppose X is modeled instead as a continuous random variable uniformly distributed between 0 and 200. Y=X-62 is still the remaining time until the next eruption, find the average remaining time until the next eruption under the same condition, E[Y|X>62]. Compare with (a), discuss your results. (13/100)

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Totally there are 6 problems.

1. (Differential Equation 20%)
   (a) (10%) Solve the second-order non-homogeneous equation $y'' - 4y = 8x^2 - 2x$.

   (b) (10%) Solve the first-order homogeneous equation $xy' = \frac{y^2}{x} + y$.

2. (Laplace Transform 15%)
   (a) (5%) Let $f(t) = e^{at}$, where a is any real number. Assuming $s > a$, please derive that

   the Laplace transform of $f(t)$ is $F(s) = \frac{1}{(s-a)}$.

   (b) (5%) Use the Laplace transform to solve $y' - 4y = 1$; $y(0) = 1$.

   (c) (5%) The inverse version of the Laplace Convolution Theorem is described as follows:

   Let $L^{-1}[F] = f$ and $L^{-1}[G] = g$, then $L^{-1}[FG] = (f * g)(t) \equiv \int_0^t f(t-\tau)g(\tau)\,d\tau$.

   Use this theorem and the result of (a) to compute $L^{-1}\left[\frac{1}{s(s-4)}\right]$.

3. (Vector Analysis 15%)
   Let **F** be a vector function in 3-dimensional space, where **F** = (a, b, c) with a=a(x,y,z), b=b(x,y,z), c=c(x,y,z).

   Also, let $\varphi(x,y,z)$ be a scalar function. In addition, the 'del' operator can be defined as follows: $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

   (a) (5%) Prove that $\nabla \times (\nabla \varphi) = \mathbf{0}$, where $\mathbf{0} = (0,0,0)$.

   (b) (5%) Prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

   (c) (5%) What is the Divergence Theorem of Gauss?

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4. (Fourier Analysis 20%)

(a) (5%) Let $f(x)=\begin{cases} 0 & \text{for } -3 \le x \le 0 \\ x & \text{for } 0 \le x \le 3 \end{cases}$, and find the Fourier series of $f(x)$ on [-3, 3].

(b) (5%) Let $f(x) = x$ on [-3, 3], and find the Fourier series of $f(x)$ on [-3, 3].

(c) (5%) The Fourier integral of $f(t)$ can be represented by $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$,

where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$.

Let $f(t)=\begin{cases} 2 & \text{for } -1 \le t \le 1 \\ 0 & \text{for } t<-1 \text{ or } t>1 \end{cases}$, and evaluate $\hat{f}(\omega)$.

(d) (5%) Plot the 'amplitude spectrum' of the function $f(t)$, which is defined in (c).

5. (Linear Algebra 15%)

Let A = $\begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

(a) (5%) Find the eigenvalues of A and their associated eigenvectors?

(b) (5%) Find an orthogonal matrix Q that can diagonalize A.

(c) (5%) Find the diagonalized matrix in (b).

6. (Complex Theory 15%)

(a) (5%) What is Cauchy's Theorem?
(You do not have to prove it, just briefly describe it.)

(b) (5%) Use Cauchy's Theorem to evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

(c) (5%) Use Residue Theorem to evaluate $\int_0^{\infty} \frac{1}{x^6+1}\, dx$.

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系(所)別：通訊工程學系碩士班　組別：通訊組　科目：工程數學　用紙第 1 頁共 1 頁

●不可使用電子計算機

1. Let A and B be both n by n matrices, choose the correct statements in the following. (10/100)
   (a). If rank(A) = n, then it is diagonalizable.
   (b). If A is invertible, then its determinant is zero.
   (c). If A has n different eigenvalues, then it is non-singular.
   (d). If A has n different eigenvalues, then its eigenvectors can serve as a basis.
   (e). If B is diagonalizable and B is similar to A, then A is also diagonalizable.

2. Let B = { $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ } and C = { $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ } be both bases of R^3, where $\mathbf{u}_1 = (1, 1, 0)^T$, $\mathbf{u}_2 = (0, 1, -1)^T$, $\mathbf{u}_3 = (1, 2, 1)^T$, $\mathbf{v}_1 = (0, 1, 1)^T$, $\mathbf{v}_2 = (1, -1, 0)^T$ and $\mathbf{v}_3 = (-1, 0, 2)^T$. (15/100)
   (a). Let S be a standard basis of R^3 and $[\mathbf{x}]_S$ means the coordinates of $\mathbf{x}$ in S. If $[\mathbf{x}]_S = (1, 1, 1)^T$, then what is $[\mathbf{x}]_B$, the coordinates of $\mathbf{x}$ in B? (5/100)
   (b). Let $[\mathbf{y}]_B = (1, 1, 1)^T$ be the coordinates of $\mathbf{y}$ in B. What is $[\mathbf{y}]_C$, the coordinates of $\mathbf{y}$ in C? (5/100)
   (c). Let A = [$\mathbf{u}_1$ $\mathbf{u}_2$ $\mathbf{u}_3$] be a 3 by 3 matrix. Please find the QR factorization of A by using Gram-Schmidt process. (5/100)

3. Find the eigenvalues and eigenvectors and the kth power of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \text{ (15/100)}$$

4. Suppose A is 3 by 4 and B is 4 by 5 and AB = 0. Prove that rank(A) + rank(B) $\leq$ 4. (10/100)

5. Suppose continuous random variables X, Y are independent and identically distributed, each with uniform probability density over [0, 1]. Conditional on the event A: X+Y>1, find the conditional joint probability density function $f_{X,Y|A}(x,y)$, conditional correlation $E[XY|A]$, conditional covariance $COV[X,Y|A]$, and the conditional correlation coefficient $\rho_{X,Y|A}$. Conditional on A, are X, Y still independent? Identically distributed? Explain. (25/100)

6. $X_1, X_2, \ldots$ is a sequence of independent, identically distributed Bernoulli random variables, each takes on the value one with probability p. The sample mean of $X_1, X_2, \ldots X_n$ is $M=(X_1+\ldots+X_n)/n$. Find the range (i.e. the set of possible values) of M, the probability mass function of M, the mean and variance of M in terms of n and p. In addition, find the probability of the event B: $X_1>X_2+X_3+\ldots+X_n$, the conditional probability $P[X_2>X_3+X_4+\ldots+X_n|B]$, and the probability of the event C: $X_1+X_2>X_3+\ldots+X_n$ assuming that n>2. (25/100)