清華大學

工程與系統科學系

92~97 學年度
工程數學考古題

國立清華大學命題紙

九十二學年度 工程與系統科學 系（所） 乙、丙、丁、戊 組碩士班研究生招生考試
3701、3902、4002、

科目 工程數學 科號 3802 共 一 頁第 一 頁 *請在試卷【答案卷】內作答

1. Given that $y_1(x) = x$ is a solution of the differential equation

$$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0.$$

Find the second solution. (15%)

2. Solve the following problem:

$$4y'' - 4y' + 17y(t) = 0. \qquad y(0) = 2, \quad y'(0) = 5.$$ (10%)

3. Find the power series solutions about point $x = 0$ of the following equation:

$$x^2 y'' + x y' - (x^2 + 1/4)\, y(x) = 0.$$ (15%)

4. Let the velocity of a fluid be described by $\mathbf{F} = 6xz\,\mathbf{i} + x^2y\,\mathbf{j} + yz\,\mathbf{k}$. Compute the rate at which fluid is leaving the unit cube. (15%)

5. (a) Prove that the eigenvalues of $k\mathbf{A}$, for any scalar k, are k times those of matrix $\mathbf{A}$. Are the corresponding eigenspaces the same? Explain. (7%)

(b) Evaluate $\iint_S \mathrm{n} \cdot \nabla \times \vec{F}\, dA$,

where $\vec{F} = xz\,\vec{i} - yz^4\vec{k}$, $S: x^2 + 4y^2 + z^2 = 4$, $x \ge 0, y \ge 0, z \ge 0$ (8%)

6. Find the steady-state temperature distribution $T(r, \theta)$ in a semicircular plate of radius 1 if

$T(1,\theta) = u_0, \quad 0 < \theta < \pi$

$T(r,0) = 0, \quad T(r,\pi) = u_0, \quad 0 < r < 1$

[in polar coordinates (r,θ), $\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}$] (15%)

7. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^\alpha}, \alpha > 1$$

(Hint: consider the contour shown in Fig. 1, $z = x + iy$) (15%)

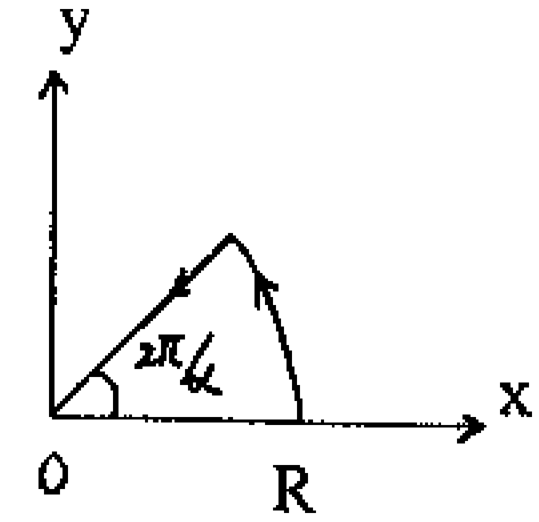

Fig. 1

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九十三學年度 工科 系（所）乙、丙、丁、戊 組碩士班入學考試

科目 工程數學 科號 3901、4002、4101、4201 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. (15%) (a) Find the particular solution of the ordinary differential equation

$$y''(x) + y(x) = \cos x \ , \qquad |x| < \infty \qquad (1)$$

(b) Find the solution of Eq.(1) with initial conditions $y(0) = 0$, and $y' = 1$.

2. (15%) If J_ν is a solution of the Bessel's equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, |x| < \infty$$

Show that

(a) $J_\nu(\alpha t)$ satisfies the equation

$$\frac{d}{dt}\left[t\frac{d}{dt}J_\nu(\alpha t)\right] + (\alpha^2 t - \nu^2/t)J_\nu(\alpha t) = 0$$

(b) $\int_0^1 tJ_\nu(\alpha t)J_\nu(\beta t)dt = 0$, where α and β are two distinct roots of $J_\nu(x) = 0$, (i.e. $J_\nu(\alpha) = J_\nu(\beta) = 0$ and $\alpha \neq \beta$).

3. (10%) Prove that vectors $\vec{u}, \vec{v}, \vec{w}$, are linearly dependent if and only if $\vec{u} \cdot \vec{v} \times \vec{w} = 0$.

4 (10%) Determine the "?" integration limits.

$$\int_0^9 \int_{z/3}^{\sqrt{z}} \int_0^{y+z} f(x,y,z)dxdydz = \int_?^? \int_?^? \int_?^? f(x,y,z)dydzdx$$

5. (10%) Solve by Fourier cosine or sine transform

$u'' - 16u = 50e^{-2x}, 0 < x < \infty$ with $u'(0) = a, u(\infty) bounded$

6. (10%) Solve the eigenvalues and eigenfunctions for

$y'' - 5y' + \lambda y = 0, (0 < x < \pi)$ as a Sturm-Liouville problem

with $y(0) = 0$ and $y(\pi) = 0$

7. (15%) Suppose that a solid right circular cylinder of radius a is of infinite extent on one side of the plane face z = 0, and that the temperature is maintained at zero along the lateral boundary, whereas the temperature distribution over the face z = 0 is prescribed as T(r,0) = f(r). Find the steady-state, axisymmetrical temperature distribution inside the cylinder.

[cylindrical coordinates (r,θ,z)

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$$]

8. (15%) Find the value of the integral

$$\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx.$$

94學年度 工程與系統科學 系（所）乙、丙、戊 組碩士班入學考試

科目 工程數學 科目代碼 3301, 3402, 3501 共 1 頁第 頁 *請在試卷【答案卷】內作答

1. Find the general solution of the inhomogeneous ordinary differential equation

$$y''(x)+y(x)=x\cos x \qquad (15\%)$$

2. Evaluate the determinant, and find the inverse matrix of matrix A

$$A=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & -1 \end{pmatrix} \qquad (10\%)$$

3. If the Laplace transform of function f(t) is F(s), i.e. F(s) = L{f(t)},
   (a) Show that

$$L\{\int_t^\infty f(u)\,du\}=\frac{1}{s}\int_0^\infty (1-e^{-st})f(t)\,dt \qquad (6\%)$$

   (b) Evaluate

$$L\{\int_t^\infty \frac{e^{-u}}{u}\,du\} \qquad (9\%)$$

4. Find a unit vector norm to the surface S given by $z=x^3y^3+y-2$ at the point (0,0,2). (10%)

5. Let $f(x,y,z)=x^2e^{-yz}$. Compute the rate of change of f in the direction $\mathbf{v}=(1,1,1)$ at point (1,0,0). (10%)

6. Evaluate the integral $\int_S F\bullet dS$ where vector $F=x\,\mathbf{i}+y\mathbf{j}+3\mathbf{k}$ and where S is the surface of the unit sphere $x^2+y^2+z^2=1$. (10%)

7. Evaluate the integral

$$\int_0^\infty \frac{\cos ax}{1-x^4}\,dx \qquad (a>0). \qquad (15\%)$$

8. Find the time-dependent temperature distribution $u(r,\theta,t)$ in a semicircular plate $0\le r\le 1$, $0\le\theta\le\pi$, given that the straight edge of the plate formed by $0\le r\le 1$, $\theta=0$ and $\theta=\pi$ is insulated, the semicircular boundary is maintained at zero temperature, and the initial temperature distribution is $u(r,\theta,0)=(1-r)\cos\theta$. (15%)

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$$\left[\text{polar coordinates}\,(r,\theta) \quad \nabla^2T=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}\right].$$

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95 學年度 工程與系統科學 系（所） 乙、丙、戊 組碩士班入學考試

科目 工程數學 科目代碼 3401 3502 3601 共 1 頁第 1 頁 *請在【答案卷卡】內作答

1. $y(x) = e^{-x}$ is a solution of

   $$(x-1)\,y'' - xy' + y(x) = 0.$$

   Using the method of reduction of order, find another linear independent solution. (10%)

2. Using Laplace transform solve the boundary value problem

   $$y'' - 2y' + y(x) = x, \qquad y(0) = 0, \quad y'(1) = -2.$$ (11%)

3. Find the general solution of the following differential equation and show the details.

   $$xy'' - 2y' + xy(x) = 0, \qquad 0 < x < \infty$$ (15%)

4. Verify the divergence theorem by working out the theorem with the giving vector function $\vec{V}$ and the volume V, where

   $\vec{V} = z^2\vec{e}_z$ and V: the cone $r \le 2z, 0 \le \theta < 2\pi, 0 \le z \le 3$

   $\vec{e}_z$ is the unit vector in the direction of z. (16%)

5. Use the method of diagonalization to obtain the general solution of the following equations:

   $x' + 2x + y = 0$

   $y' + x + 2y + z = 0$ where primes denote d/dt. (16%)

   $z' + y + 2z = 0$

6. Solve the diffusion equation

   $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 < x < 1, t > 0$$

   subject to the initial condition

   $$u(x, 0) = 1 \qquad \text{for } 0 < x < 1,$$

   and the boundary conditions

   $$u(0, t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -h\,u(1, t) \qquad \text{for } t > 0, \quad h > 0.$$ (16%)

7. Compute

   $$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 3x + 2}\,dx$$ (16%)

1. You are required to use residues to find the value of the integral

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} \qquad (-1<a<1\ ).$$

(15%)

2. Suppose that the steady-state temperature T in a solid right circular cylinder of radius a possesses axial symmetry, and hence is of the form $T=T(r,z)$, where r is distance from the z axis. The temperature T then must satisfy the equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})+\frac{\partial^2 T}{\partial z^2}=0$$

inside the cylinder. Suppose that the faces $z=0$ and $z=L$ of the solid right circular cylinder are maintained at temperature zero, and that the temperature distribution along the lateral boundary $r=a$ is prescribed as $T(a,z)=f(z)$. Find the resultant steady-state temperature distribution inside the cylinder.

(15%)

3. Find the general solution of the following differential equation

$$xy'-16-2y(x)-2x^{-1}+15x^{-2}=0.$$

(10%)

4. Obtain, and compare the solution to

(a) $y''+2y'+5y(t)=0, \qquad y(0)=0,\ y'(0)=1;$

(b) $y''+2y'+5y(t)=\delta(t), \qquad y(0)=0,\ y'(0)=0.$

where $\delta(t)$ is the Dirac delta function (unit impluse function)

$$\delta(t)=\begin{cases}\infty & \text{if } t=0\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \int_0^{\infty}\delta(t)dt=1.$$

(10%)

5. Solve the initial value problem of the first-order system

$$\begin{cases} x' = x + y \\ y' = x + y + e^{2t} \\ x(0) = y(0) = 0 \end{cases} .$$

(8%)

6. Let S be the surface (with outer unit normal $\hat{n}$) of the region R bounded by the planes $z = 0$, $y = 0$, $y = 4$ and the paraboloid $z = 1 - x^2$. Compute $\iint_S \vec{F} \cdot \hat{n}\, dS$, given

$\vec{F} = (x + \sin y)\hat{i} + (2y + \cos z)\hat{j} + (3z + 4e^x)\hat{k}$.

(7%)

7. Find the surface of the torus generated by revolving the circle $(x-a)^2 + z^2 = b^2$ in xz-plane around z-axis with $b < a$.

(8%)

8. Express the periodic function $f(x) = |\cos x|$ in its Fourier series FS $f = \sum_{n=-\infty}^{\infty} c_n \exp(i2nx)$. Work out $c_n = ?$

(7%)

9. Use power series method to solve

$$y'' + 12y' + x^3 y(x) = 0.$$

Find at least five terms of the general solution.

(10%)

10. Find the inverse Laplace transform of

$$\frac{e^{-5s}}{s(s^2 + 12)}$$

(10%)

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97學年度工程與系統科學系乙、丙組、先進光源科技學位學程乙組及核子工程與科學研究所甲組碩士班入學考試

科目工程數學 科目代碼 2901、3001、3201、3101 共 2 頁第 1 頁 *請在【答案卷卡】內作答

1. (a) Evaluate

$$\int_{(0,0)}^{(2,1)} (5y^3 + 20x^4y^2)dx + (15xy^2 + 8x^5y - 3)dy$$

along the path $x^4 - 6xy^3 = 4y^2$.

(5%)

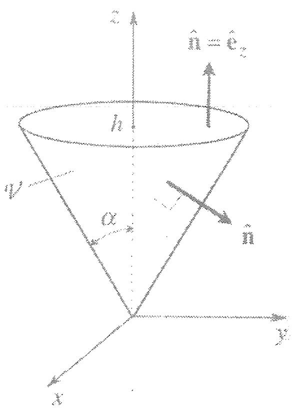

Figure The cone V

(b) Let $\vec{v} = rz^2\hat{e}_z$, which is given in cylindrical coordinates (r, θ, z). Evaluate $\oint_S \hat{n} \cdot \vec{v} dA$ where S is the surface of the cone V (see figure).

(5%)

2. (a) Find an orthonormal set of the linear independent set {(2,0,0), (1,1,0), (3,3,3)} using Gram-Schmidt orthogonalization process. (5%)

(b) Matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{pmatrix}$ can be expressed as $Q^t DQ$ where D is a diagonal matrix and Q is an orthogonal matrix. Find D and Q. (5%)

3. Find the *harmonic* function $u(x, y)$ in the semi-infinite strip $0 < x < \pi$, $y > 0$ such that

$$u(0, y) = u(\pi, y) = 0 \qquad (y > 0),$$
$$u(x, 0) = 1 \qquad (0 < x < \pi),$$

and $|u(x, y)| < M$, where M is some constant.

(10%)

4. Determine the residue of each of the following functions at each singularity:

(a) $\tan z$,

(b) $\dfrac{\sin z - z}{z^6}$,

(c) $z\, e^{1/z}$.

(10%)

5. Solve by Fourier transform

$$u'''' + ku = w(x),$$

where k is constant and w(x) can be expanded in a Fourier integral, and $u(x), u'(x), u''(x), u'''(x) \to 0$, as $x \to \pm\infty$.

Hint: the Fourier transform of $f(x) = e^{-a|x|/\sqrt{2}} \sin\left(\frac{a}{\sqrt{2}}|x| + \frac{\pi}{4}\right)$ $(a > 0)$ is $\frac{2a^3}{\omega^4 + a^4}$.

(10%)

6. Find the weighting function of the following equation to become a SLP (Sturm-Liouville Problem) -type equation, $(1-x^2)y'' - xy' + \lambda y = 0$.

(5%)

7. Find the general solution $y(x)$ of the following differential equation

$$x^2 y'' - 2xy' + 2y = x \ln x.$$

(Hint: let $x = e^t$.)

(15%)

8. (a) Prove the following relations between Laplace transforms

$$L\{y'(t)\} = sL\{y(t)\} - y(0),$$

$$L\{t\, y(t)\} = -\frac{d}{ds} L\{y(t)\}.$$

(b) Solve the following problem using Laplace transform

$$t\, y'' + 2t\, y' + 2y = 0; \qquad y(0) = 0.$$

(15%)

9. Prove the recurrence relations satisfied by Legendre polynomials

$$(k+1)P_{k+1}(x) - (2k+1)\, x\, P_k(x) + k\, P_{k-1}(x) = 0. \qquad k = 1, 2, 3, \ldots$$

(Hint : A generating function of the Legendre polynomials is

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Differentiate the above equation once with respect to t.)

(15%)

清華大學

光電工程研究所

92~97 學年度
工程數學考古題

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九十二學年度 光電工程研究 系(所) ＿＿＿組碩士班研究生招生考試

科目 工程數學 科號 2501 共 二 頁第 一 頁 *請在試卷【答案卷】內作答

1. Solve $y''+2y'+y=0$ for the general solution y(x). (5%)

2. Derive the general solution y(x) for the equation
$y'+f(x)y=g(x)$. (10%)

3. Write a partial differential equation that has the solution
$y(x,t) = (4x+8)\cdot(-t/4+1)$. Note that $\partial y/\partial x$ and $\partial y/\partial t$ must be included in the equation. Also note that you have to give a reason for bringing up your answer, i.e., you have to show how you derive your answer. (15%)

4. The velocity of a rotating particle is given by a vector $\mathbf{V} = \boldsymbol{\omega}\times\mathbf{r}$, where $\boldsymbol{\omega}$ is a constant vector. Find the value of $\nabla\times\mathbf{V}$. (5%)

5. Find the surface integral of the vector function $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ over that portion of the surface $z = xy+1$, which covers the square $0 \le x \le 1$, $0 \le y \le 1$ in the xy plane. (10%)

6. Find all Taylor or Laurent series representations with center $z_0 = 1$, and their corresponding precise region of convergence of the function $f(z) = \sinh z/(z-1)^2$. (10%)

7. Decide whether the following matrices are positive definite, negative definite, or indefinite? (a) $\begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{pmatrix}$ (15%)

8. Assume that the coefficient of complex Fourier series of a periodic function $f_1(t)$ with period = 4 shown in Fig. 1 is c_n, where $-\infty < n < \infty$.

(a) Find the complex Fourier series representation for a periodic function $f_2(t)$ with the same period (= 4) shown in Fig. 2 in terms of c_n. (8%)

(b) Find the Fourier series representation of the solution y(t) in terms of c_n if

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+y(t)=f_2(t),\quad -\infty < t < \infty. \qquad (7\%)$$

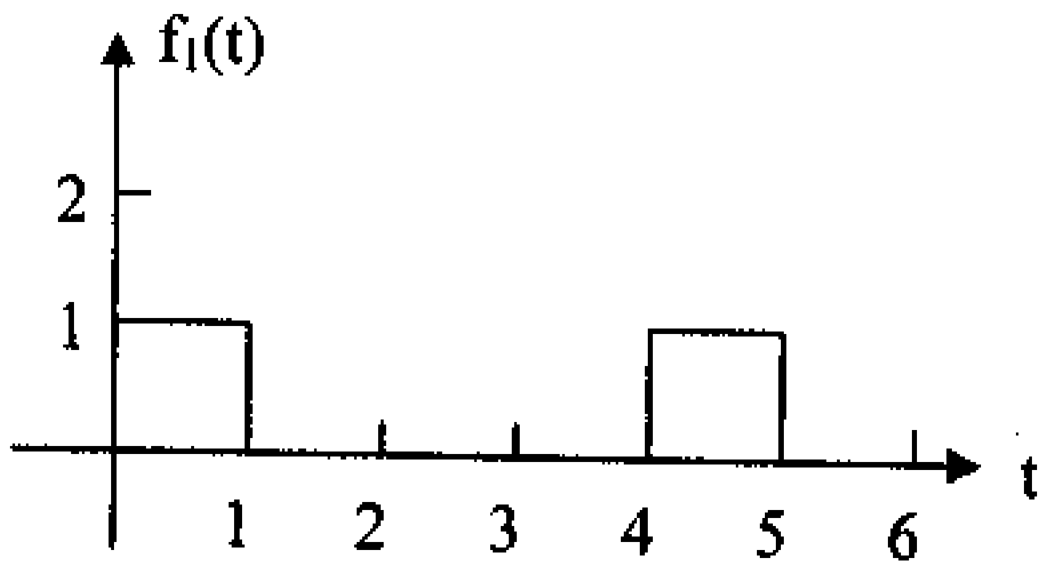


Fig. 1

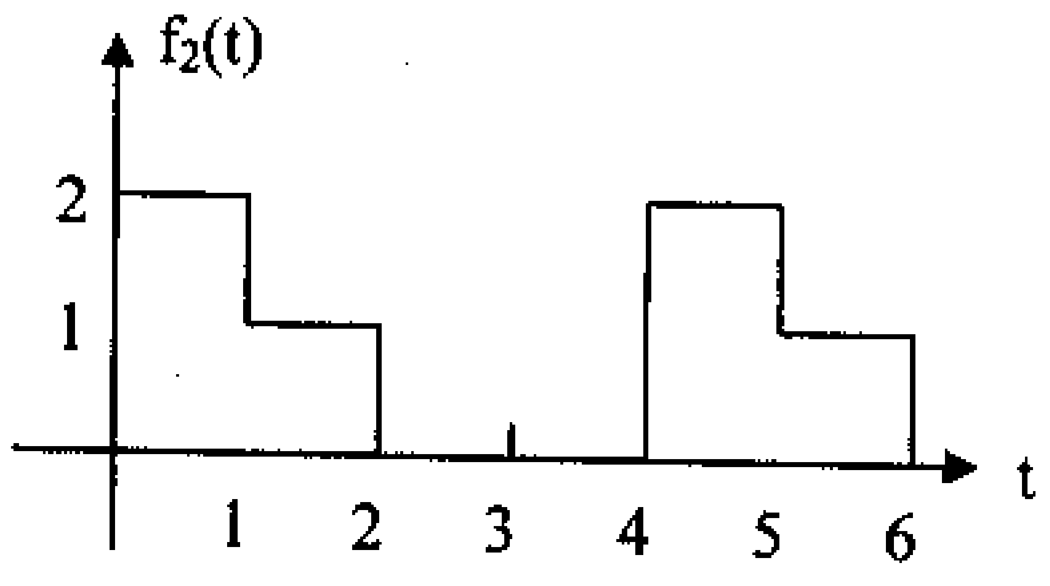


Fig. 2

9. The capacitor voltage $v_o(t)$ in a series RLC circuit satisfies the following differential equation:

$$\frac{1}{2}\frac{d^2v_o(t)}{dt^2}+\frac{3}{2}\frac{dv_o(t)}{dt}+v_o(t)=v_i(t),\qquad \left.\frac{dv_o(t)}{dt}\right|_{t=0^+}=2,\ v_o(0^+)=1.$$

Use Laplace transform to find the capacitor voltage $v_o(t)$ for $t > 0$ if the voltage source $v_i(t) = e^{-3t}u(t)$. (15%)

1. (5%)$\mathcal{L}$ represents the Laplace Transform operator.

$$\mathcal{L}\Big(t\cos(2t)\Big) = (\underline{\quad(1)\quad} - 4)\cdot \underline{\quad(2)\quad}$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A) s^2; (B) s^{-2}; (C) $(s-1)$; (D) $(s-1)^2$; (E) $(s-1)^{-2}$; (F) (s^2-1); (G) (s^2-2); (H) $(s-2)^2$; (I) $(s-2)^{-2}$; (J) $(s^2+4)^{-2}$; (K) $(s^2+4)^2$.

2. (5%)$\mathcal{L}^{-1}$ represents the inverse Laplace Transform operator.

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \underline{\quad(1)\quad}\Big(\sin(\omega t) - \underline{\quad(2)\quad}\cdot\cos(\omega t)\Big)$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A)2ω; (B)$\frac{1}{2\omega}$; (C)$2\omega^2$; (D)$\frac{1}{2\omega^2}$; (E)$2\omega^3$; (F)$\frac{1}{2\omega^3}$; (G)ωt; (H)$(\omega t)^2$; (I)$(\omega t)^3$; (J)$2\omega^2 t$; (K)$2\omega^2 t$.

3. (5%)'$*$' represents the convolution operator.

$$(e^{-t} - e^{-2t}) * e^{-t} = \underline{\quad(1)\quad} + (t-1)\underline{\quad(2)\quad}$$

Please find (1) and (2) from the following. Both have to be correct to receive full grade.
(A)e^t; (B)$e^{(t-1)}$; (C)e^{-t}; (D)$e^{-(t-1)}$; (E)e^{-2t}; (F)$e^{-2(t-1)}$; (G)e^{2t}; (H)$e^{2(t-1)}$; (I)e^{-3t}; (J)$e^{-3(t-1)}$; (K)e^{3t}; (L)$e^{3(t-1)}$; (M)t; (N)$(t-1)$; (O)$\frac{1}{t-1}$; (P)$(t-1)^2$; (Q)$(t-1)^3$.

4. (5%)Please identify all the even functions in the following. Full grade will be given only if all answers are correct. (A)e^x; (B)$e^{(x^2)}$; (C)$\sin(nx)$; (D)$x\sin(x)$; (E)$\frac{\cos(x)}{x}$; (F)$ln(x)$; (G)$\sin(x^2)$; (H)$\sin^2(x)$.

5. (5%)Which of the following collections of vectors are linearly independent in R^3? R^3 represents a Euclidean vector space. (A)$(1,0,0)^T,(0,1,1)^T,(1,0,1)^T$; (B)$(1,0,0)^T,(0,1,1)^T,(1,0,1)^T,(1,2,3)^T$; (C)$(2,1,-2)^T,(3,2,-2)^T,(2,2,0)^T$; (D)$(2,1,-2)^T,(-2,-1,2)^T,(4,2,-4)^T$;(E)$(1,1,3)^T,(0,2,1)^T$.

6. (7%) The standard 2nd-order mass-damper-spring system can be expressed by the differential equation $m\ddot{x}+b\dot{x}+kx=F(t)$, where x is the displacement of the proof mass, b is the damping coefficient, k is the spring constant, and $F(t)$ is the externally applied force. The equation can be re-written in another form as $\ddot{x}+2\xi\omega_n\dot{x}+\omega_n^2 x=F(t)/m$, where ξ is the damping ratio and ω_n is the natural frequency defined as $\omega_n=\sqrt{\dfrac{k}{m}}$. Now that the applied force $F(t)$ is a unit-step function $u(t)$ and $0<\xi<1$. Determine the corresponding particular solution from the following answers (note: $\omega_d=\omega_n\sqrt{1-\xi^2}$):

(1) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(cos\,\omega_d t+sin\,\omega_d t\right)\right]$

(2) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(1+\omega_n t\right)\right]$

(3) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(\dfrac{1}{\sqrt{1-\xi^2}}+\omega_n t\right)\right]$

(4) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}-e^{-\xi\omega_d t}\right]$

(5) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}-\dfrac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_d t}\right]$

(6) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(cos\,\omega_d t+\dfrac{\xi}{\sqrt{1-\xi^2}}sin\,\omega_d t\right)\right]$

(7) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(\dfrac{1}{\sqrt{1-\xi^2}}cos\,\omega_d t+sin\,\omega_d t\right)\right]$

(8) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(\dfrac{\xi}{\sqrt{1-\xi^2}}cos\,\omega_d t+sin\,\omega_d t\right)\right]$

(9) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(c_1\,cos\,\omega_d t+c_2\,sin\,\omega_d t\right)\right]$, c_1 and c_2 are arbitrary constants.

(10) $x(t)=\dfrac{1}{k}\left[1-e^{-\xi\omega_n t}\left(c_1+c_2\omega_n t\right)\right]$, c_1 and c_2 are arbitrary constants.

7. (6%) The differential equation $axy''+y'+y=0$ ($0<x<\infty$, a is an unknown constant) has two linear independent solutions expressed in power series: $y_1(x)=1-x+\dfrac{x^2}{8}-\dfrac{x^3}{168}+\dfrac{x^4}{6720}-\dots$, $y_2(x)=x^{2/3}-\dfrac{x^{5/3}}{5}+\dfrac{x^{8/3}}{80}-\dfrac{x^{11/3}}{2640}+\dots$.
Please determine the value of a that leads to these two solutions. (1) a = -1 (2) a = 1 (3) a = -2 (4) a = 2 (5) a = -3 (6) a = 3 (7) a = -4 (8) a = 4 (9) a = -1/2 (10) a = 1/2.

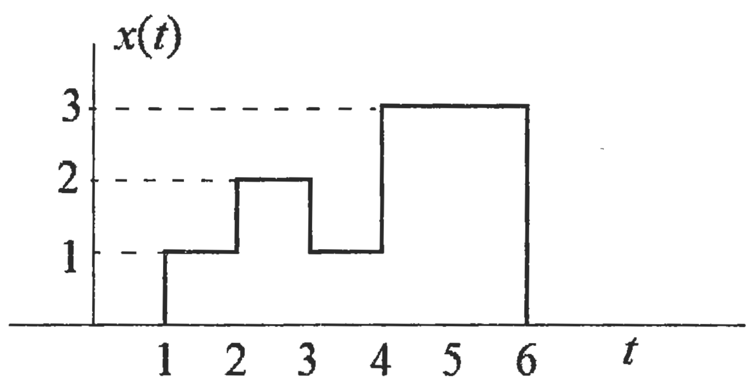
8. (7%) Determine the general solution of the differential equation $y'=y^2-xy+1$, which has a particular solution $Y(x)=x$ by inspection (note: C is an arbitrary constant).

(1) $y(x)=x+\dfrac{2e^{-x^2/2}}{C-3\int e^{x^2/2}dx}$ (2) $y(x)=x+\dfrac{e^{x^2/2}}{C-\int e^{x^2/2}dx}$ (3) $y(x)=x+\dfrac{e^{-x^2/2}}{C-\int e^{-x^2/2}dx}$ (4) $y(x)=x+\dfrac{2e^{x^2/2}}{C+\int e^{-x^2/2}dx}$ (5) $y(x)=x-\dfrac{e^{-x^2/2}}{C-2\int e^{x^2/2}dx}$

(6) $y(x)=x-\dfrac{e^{x}}{C-\int e^{x}dx}$ (7) $y(x)=x+\dfrac{2e^{x}}{C+\int e^{-x}dx}$ (8) $y(x)=x+\dfrac{2e^{x}}{C+3\int e^{-x}dx}$

(9) $y(x)=x+\dfrac{e^{-x}}{C+\int e^{-x}dx}$ (10) $y(x)=x+\dfrac{2e^{x}}{C+2\int e^{-x}dx}$.

9. (7%) Find the Fourier transform of the function $x(t)$ shown below.



10. (10%) Solve for u(x,t) that satisfies $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the following conditions

$u(0,t) = u(1,t) = 0$ for all t

$u(x,0) = \sum_{n=1}^{7} \frac{1}{n} \sin n\pi x$, $\frac{\partial u}{\partial t}\Big|_{t=0} = 0$ for $0 < x < 1$

You need to show how you derive your answer. Partial points will be deducted for not writing your derivation.

11. (4%) Find a scalar function $f(x,y,z)$ such that $\nabla f = 6x\vec{i} + 2\vec{j} + 2z\vec{k}$. No need to write down the derivation. Just giving your answer is OK.

(12%) Then, choose an answer for each of the following integrals along the specified paths: (No need to write down the derivation. Just pick up the correct value for each integral.)

$\int_C \left(6x\vec{i} + 2\vec{j} + 2z\vec{k}\right) \bullet d\vec{r}$ =(a) 0 (b) 2 (c) 2π (d) 4π (e) 4 (f) 6 (g) 2.5π (h) 10 (i) none of the above

$\int_D \left(6x\vec{i} + 2\vec{j} + 2z\vec{k}\right) \bullet d\vec{r}$ =(a) 0 (b) 2 (c) 2π (d) 4π (e) 4 (f) 6 (g) 2.5π (h) 8 (i) none of the above

The absolute value of $\oint_E \left(yz\vec{i} + 6xz^5\vec{j} - xy^2z\vec{k}\right) \bullet d\vec{r}$ is equal to (a) 0 (b) 2π (c) 3π (d) 5π (e) 8π (f) 11π (g) 13π (h) π (i) none of the above

Here, C is the path from the point (0,0,0) to (1,1,1) following a straight-line segment.

D is the path first from the point (0,0,0) to $(0,\frac{1}{2},0)$ following a straight-line segment, and then from $(0,\frac{1}{2},0)$ to (1,1,1) again following a straight-line segment.

E is the path along the circle: $x^2 + y^2 = 1,\quad z = 1$.

12. (12%) $\int_0^\infty \frac{\sin x}{x} dx =$ (a) 0.4π (b) π (c) 2π -4 (d) 2.5π -5 (e) 0.6π (f) 0.8π

(g) π -2 (h) 0.5π (i) none of the above. (You may use the residue theorem.)

13. (10%) Evaluate the integrals along the path C that is the counterclockwise circle with $|z| = 3$.

(a) $\oint_C \frac{z^2 - 1}{z^2 + 1} e^z dz$

(b) $\oint_C \frac{\sinh 3z}{(z^2 + 1)^2} dz$

For problems 1~5, both correct answers and detailed works are required.

1. (5 %) Find the sine half-range expansion of $f(x)$

$$f(x)=\begin{cases} \dfrac{2k}{L}x & 0<x<\dfrac{L}{2} \\ & \text{if} \\ \dfrac{2k}{L}(L-x) & \dfrac{L}{2}<x<L \end{cases}$$

(A) $\dfrac{4k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x+\dfrac{1}{3^2}\sin\dfrac{3\pi}{L}x+\dfrac{1}{5^2}\sin\dfrac{5\pi}{L}x+...\right)$ (B) $\dfrac{4k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x-\dfrac{1}{2^2}\sin\dfrac{2\pi}{L}x+\dfrac{1}{3^2}\sin\dfrac{3\pi}{L}x-...\right)$

(C) $\dfrac{8k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x-\dfrac{1}{3^2}\sin\dfrac{3\pi}{L}x+\dfrac{1}{5^2}\sin\dfrac{5\pi}{L}x-...\right)$ (D) $\dfrac{8k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x+\dfrac{1}{2^2}\sin\dfrac{2\pi}{L}x+\dfrac{1}{3^2}\sin\dfrac{3\pi}{L}x+...\right)$

(E) $\dfrac{4k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x-\dfrac{1}{3^2}\sin\dfrac{2\pi}{L}x+\dfrac{1}{5^2}\sin\dfrac{3\pi}{L}x-...\right)$ (F) $\dfrac{6k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x-\dfrac{1}{3^2}\sin\dfrac{2\pi}{L}x+\dfrac{1}{5^2}\sin\dfrac{3\pi}{L}x-...\right)$

(G) $\dfrac{2k}{\pi^2}\left(\dfrac{1}{1^2}\sin\dfrac{\pi}{L}x-\dfrac{1}{3^2}\sin\dfrac{2\pi}{L}x+\dfrac{1}{5^2}\sin\dfrac{3\pi}{L}x-...\right)$ (H) none of the above

2. (5 %) Find the Fourier transform of $f(x)$

$$f(x)=e^{-|x+3|}-2e^{-|x|}$$

(A) $\dfrac{1}{\sqrt{2\pi}(w+1)}\left(e^{-i3w}-2\right)$ (B) $\dfrac{2}{\sqrt{2\pi}(w+1)}\left(e^{i3w}-2\right)$ (C) $\dfrac{2}{\sqrt{2\pi}(w^2+1)}\left(e^{-i3w}-2\right)$

(D) $\dfrac{2}{\sqrt{2\pi}(w^2+1)}\left(e^{i3w}-2\right)$ (E) $\dfrac{1}{\sqrt{2\pi}(w^2-1)}\left(e^{i3w}-2\right)$ (F) $\dfrac{1}{\sqrt{2\pi}(w^2+1)}\left(e^{i3w}-2\right)$

(G) $\dfrac{1}{\sqrt{2\pi}(w^2-1)}\left(e^{-i2w}-3\right)$ (H) none of the above

3. (5 %) Find the inverse Laplace transform of

$$F(s)=\frac{1}{s(s^2+\omega^2)}$$

(A) $\frac{1}{w^2}(1-\sin wt)$ (B) $\frac{1}{w^2}(1+\cos wt)$ (C) $\frac{1}{w^2}(1-\cos wt)$ (D) $\frac{1}{w}(1-\sin wt)$

(E) $\frac{1}{w}(1+\cos wt)$ (F) $\frac{1}{w}(1+\tan wt)$ (G) $\frac{1}{w}(1-\tan wt)$ (H) none of the above

4. (10 %) Use Laplace transform to solve

$$xy''+(1-x)y'+ky=0$$

(A) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^{-k}e^{-t}]$ (B) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^k e^t]$ (C) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^k e^{-t}]$ (D) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^k e^{-t}]$

(E) $y=\frac{e^{-t}}{k!}\frac{d^k}{dt^k}[t^k e^{-t}]$ (F) $y=\frac{e^{-t}}{k}\frac{d^k}{dt^k}[t^k e^{-t}]$ (G) $y=\frac{e^k}{t!}\frac{d^k}{dt^k}[t^k e^{-t}]$ (H) none of the above

5. (10 %) Use Method of Frobenius to solve the general solution of

$$y''+\frac{1}{2x}y'+\frac{1}{4x}y=0$$

(A) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n-\frac{1}{2}}$ (B) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}$

(C) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}$ (D) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{n+\frac{1}{2}}$

(E) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)!}x^{n-\frac{1}{2}}$ (F) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n-1)!}x^{n-\frac{1}{2}}$

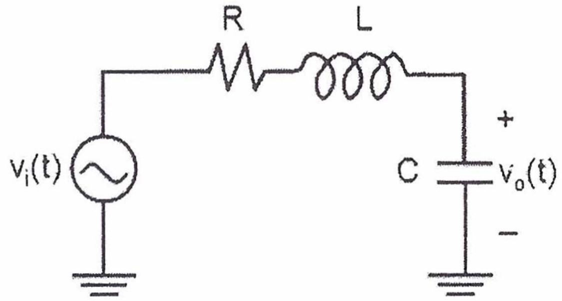
(G) $y=c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!}x^n+c_2\sum_{n=0}^{\infty}\frac{}{(2n-1)!}x^{n+\frac{1}{2}}$ (H) none of the above

(c_1 and c_2 are arbitrary constants)

6. (a) (3%)The R-L-C network as shown has a sinusoidal input $v_i(t) = \sin(\omega_o t)$, and the output voltage across the capacitor is described by the differential equation:

$$\frac{d^2 v_o(t)}{dt^2} + 30\frac{dv_o(t)}{dt} + 22500 v_o(t) = v_i(t)$$

where the coefficients are determined by the value of each passive component.



You are required to calculate the input frequency ω_o that will cause the output $v_o(t)$ to have an exact 90° phase delay with respect to the input $v_i(t)$, as the output reaches its steady state (namely, the particular solution of the differential equation).

(b)(4%) By using the differential operator $D^n = \frac{d^n}{dx^n}$, the differential equation

$$\frac{d^6 y}{dx^6} + 2\frac{d^5 y}{dx^5} + 9\frac{d^4 y}{dx^4} - 2\frac{d^3 y}{dx^3} - 10\frac{d^2 y}{dx^2} = sin(3x) + 3x^2 + xe^{-x}$$

is re-written as

$$(D^2 + 2D + 10)(D^4 - D^2)y = sin(3x) + 3x^2 + xe^{-x}.$$

Please determine the correct representation of the particular solution y_p for solving, and you do not have to solve the coefficients in it.

7. (5%) Solve the differential equation $cos\,x \cdot dx + (sin\,x + cos\,y - sin\,y) \cdot dy = 0$.

8. (8%) Solve the differential equation $x^3 \frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} - 9xy = 1 \quad (x > 0)$.

9. (10%) Evaluate the integral $\oint_C e^{\frac{1}{z^2}} dz$ where C:$|z|=4$ **counterclockwise**.

10. (10%) Find the eigenvalues and corresponding normalized eigenvectors (norm equals to 1) for the matrix $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 4 & 2 & 5 \end{bmatrix}$.

11. The position $\bar{r}$ of a particle of mass m=1 at time t is described as (all physical quantities are in SI units):

$$C: \ \bar{r}(t) = \frac{t^2}{\sqrt{2}}\bar{i} + (t+1)\bar{j} + \frac{t^3}{3}\bar{k}, \ t=[0,1].$$

(a) (4%) Let V and W denote the average speed (a scalar) and work done to move the particle from t=0 to t=1, respectively. Choose the correct answer of (V,W) from the following:

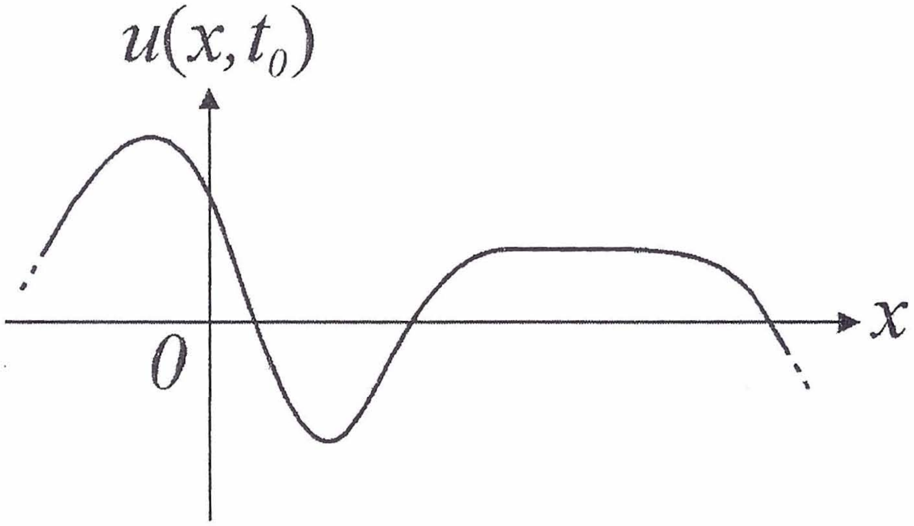
(a) (1,2); (b) (2,1); (c) $\left(\frac{1}{3},\frac{1}{2}\right)$; (d) $\left(\frac{1}{2},\frac{2}{3}\right)$; (e) $\left(\frac{3}{2},\frac{4}{3}\right)$; (f) $\left(\frac{4}{5},\frac{3}{2}\right)$; (g) (1,1); (h) $\left(1,\frac{1}{2}\right)$; (i) $\left(\frac{4}{3},\frac{5}{2}\right)$; (j) $\left(\frac{1}{2},\frac{1}{3}\right)$; (k) $\left(\frac{4}{3},\frac{3}{2}\right)$; ($l$) none of the above.

(b) (3%) If there exists an electric field $\bar{E}(x,y,z) = y\cdot\cos(z)\bar{i} + x\cdot\cos(z)\bar{j} - xy\cdot\sin(z)\bar{k}$. What is the work W_E done by the field $\bar{E}$ to move the particle of charge $q=\sqrt{2}$ along the specified path C: $\bar{r}(t)$, t=[0,1]?

(a) sin(2); (b) 1; (c) $\sin\left(\frac{1}{3}\right)$; (d) $2\sin\left(\frac{2}{3}\right)$; (e) $\sqrt{2}\cos\left(\frac{1}{3}\right)$; (f) $\sqrt{2}\sin\left(\frac{2}{3}\right)$; (g) $\sqrt{2}$; (h) $\frac{\sqrt{3}}{2}$; (i) $\frac{2}{3}\cos\left(\frac{2}{3}\right)$; (j) $\frac{1}{2}\cos\left(\frac{1}{3}\right)$; (k) $2\cos\left(\frac{1}{3}\right)$; ($l$) none of the above.

12. The motion of a string is governed by the partial differential equation (PDE): $u_{tt}=c^2u_{xx}$; where $u(x,t)$ is the displacement of the particle at position x and time t, c is a real constant, the subscripts tt, xx denote $\partial^2/\partial t^2$, $\partial^2/\partial x^2$, respectively.

(a) (5%) The following figure shows a section of the string at some instant $t=t_0$, please roughly sketch the force vectors imposing on the illustrated string section.



(b) (8%) Let the string has a finite length L ($0\le x\le L$), and the two ends slide vertically without friction, i.e. boundary conditions (BCs) are: $u_x(0,t)=u_x(L,t)=0$, where the subscript x denotes $\partial/\partial x$. One can derive discrete modes $u_n(x,t)=X_n(x)\cdot T_n(t)$ (functions satisfying the PDE and BCs) by using the method of separation of variables. Please sketch the spatial profile $X_n(x)$ for the lowest three (nontrivial) modes.

(c) (5%) In the presence of initial conditions (ICs): $u(x,0)=f(x)$, $u_t(x,0)=g(x)$, one usually expands the solution in terms of the modes: $u(x,t)=\sum_n \{A_n\}u_n(x,t)$, where $\{A_n\}$ is(are) the coefficient(s) for mode $u_n(x,t)$, then substitutes ICs to retrieve $\{A_n\}$. Although the principle of superposition works for the PDE of this problem ($u_{tt}=c^2u_{xx}$), it could fail in some other PDEs. Please specify those of the following PDEs for which superposition does NOT apply.

(a) $u_{tt}=p(x)\cdot u_{xx}$; (b) $u_{tt}=p(x)\cdot u_{xx}+q(x,t)$; (c) $u_{tt}=u_{xx}+u_{xt}$; (d) $u_{tt}=p^2(x)\cdot u_{xx}+u_{xt}$; (e) $u_{tt}= u\cdot u_t+u_x$; (f) $u_t=\exp[u_x]+u_{tt}$; (g) $u_{ttx}=p(x,t)\cdot u_{xxt}$; (h) $u_{tt}= \exp[p(x,t)]\cdot u_{xx}+u$.

國 立 清 華 大 學 命 題 紙

96 學年度 電機領域聯合招生 系（所） ______ 組碩士班入學考試

科目 工程數學 A 科目代碼 9902 共 5 頁第 1 頁 *請在【答案卷卡】內作答

(1)至(9)爲選擇題，包含單選及複選型態. 完全答對始給分，答錯倒扣該題之 50%.

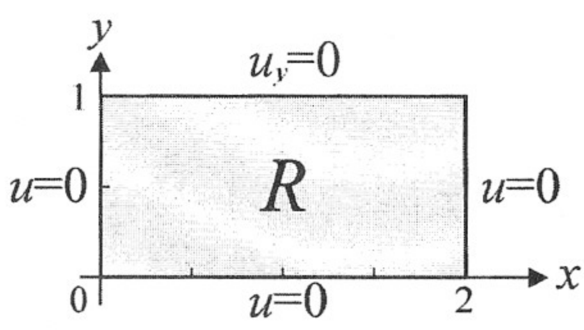
The temperature distribution $u(x,y,t)$ in a rectangular region R:$\{0\le x\le 2,\ 0\le y\le 1\}$ is governed by a partial differential equation (PDE): $u_t=\alpha^2(u_{xx}+u_{yy})$, where α is a real constant, and the subscripts t, xx, yy denote partial derivatives $\partial/\partial t$, $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, respectively.

(1) (6%) If the three ordinary differential equations (ODEs) derived by separation of variables: $u(x,y,t)=X(x)\cdot Y(y)\cdot T(t)$ are: $X''+k^2X=0$, $Y''+h^2Y=0$, $\dot{T}+\frac{1}{\tau}T=0$, what is the relation among the three eigenvalues k, h, τ?

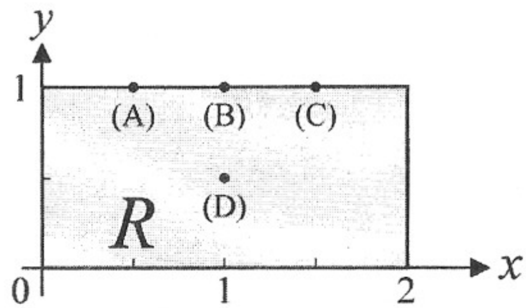
(A) $k^2-h^2=\frac{\alpha^2}{\tau}$; (B) $k^2+h^2=\frac{\tau}{\alpha^2}$; (C) $k^2+h^2=\frac{\alpha^2}{\tau}$; (D) $k^2-h^2=\frac{\alpha^2}{\tau^2}$;

(E) $k^2+h^2=\frac{1}{\tau\alpha^2}$.

(2) (6%) The boundary conditions (BCs) are specified as follows ($u_y\equiv\partial u/\partial y$):



Let the fundamental mode $u_{min}(x,y,t)$ be the solution to the PDE and BCs, which corresponds to the maximum of eigenvalue τ. What is the position $(x_0,\ y_0)\in R$ where the fundamental mode has peak magnitude, i.e. $|u_{min}(x_0, y_0, t)|$ is maximum at any time t?



(A) $\left(\frac{1}{2},1\right)$; (B) $(1,1)$; (C) $\left(\frac{3}{2},1\right)$; (D) $\left(1,\frac{1}{2}\right)$;

(E) none of the above.

The idea of d'Alembert's solution is to transform the wave equation into canonical form which can be solved more easily. Similar approaches can be applied to some other cases. Consider the linear first order PDE $u_x + 2u_y + u = 0$.

(3) (8%) which of the following transformations can reduce it to an ODE?

(A) $\Phi = x + y$, $\Psi = x - 2y$; (B) $\Phi = x + y$, $\Psi = 2x - y$;

(C) $\Phi = x + y$, $\Psi = x - y$; (D) $\Phi = x + 2y$, $\Psi = x - y$;

(E) none of the above.

(4) (8%) solve $u_x + 2u_y + u = 0$ with the condition $u = 1$ when $x + y = 1$. Calculate $u(x = 2, y = 2) = ?$

(A) e^{-1}; (B) e^{1}; (C) $e^{-2/3}$; (D) $e^{2/3}$; (E) none of the above.

(5) (8%) Let A be an n-by-n matrix, and $x \in \Re^n$. Please find the correct statements from the following:

(A) $\|x\|_1 \le n\|x\|_\infty$; (B) $\|x\|_2 > \sqrt{n}\|x\|_\infty$; (C) $\|x\|_\infty \le \|x\|_2$; (D) $\|Ax\|_\infty > \sqrt{n}\|A\|_2\|x\|_\infty$;

(E) $\frac{1}{\sqrt{n}}\|A\|_2 \le \|A\|_\infty \le \sqrt{n}\|A\|_2$

(6) (8%) Evaluate the integral $\int_R e^{-z}dz$ along the path R that is the positive x axis (from the origin to the point $(\infty, 0)$, where ∞ represents some number approaching infinity). (A) 0 (B) $2\pi j$ (C) 1 (D) $e^{-0.5\pi}$ (E) 0.5π

(7) (10%) Evaluate the integral $\oint_C \left[z^2 - \mathrm{Re}(z)\right]dz$ along the path C that is counterclockwise circle with $|z| = 2$.

(A) $-2\pi i$; (B) $-4\pi i$; (C) $2i - 4\pi$; (D) $2 - 4\pi i$; (E) $4 + 2\pi i$.

(8) (10%) $\oint_C \frac{e^{-z}}{\cos 4z}dz$ = (A) $2\pi j\, e^{-\frac{\pi}{8}}$; (B) 0; (C) $\pi j \sinh(\frac{\pi}{8})$; (D) $2\pi j\, e^{0.125\pi}$;

(E) $\pi j\, e^{\frac{\pi}{8}}$

(Here C is the counterclockwise path along the circle centered at the origin with a radius of unity.)

(9) (10%) Pick the correct statements regarding the matrix $\mathbf{A}=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$.

(A) There are three sets of linearly independent eigenvectors associating with three distinct eigenvalues. (B) The determinant of this matrix is 45. (C) $\begin{bmatrix}2\\1\\0\end{bmatrix}$ is one of the eigenvectors. (D) 5 is an eigenvalue. (E) The homogeneous linear system $\mathbf{Ax}=0$ has no non-trivial solution.

(10)至(13)爲計算題，請在答案卷上寫出該題之推導過程.

For problems 10 and 11, please write down your work and pick the correct answer codes (ex. A1 x C5) from the list.

(10) (6%) Find the inverse Laplace transform $f(t)$ of $L^{-1}\left\{\frac{(s-1)^n}{s^{n+1}}\right\}$.

ANS： (10-1) × (10-2)

(11) (6%) Suppose： $F(w)=\begin{cases}1 & , \quad |w|\le W\\ 0 & \quad\ \ |w|\ge W\end{cases}$ Find the inverse Fourier transform $f(x)$

ANS： (11-1) × (11-2)

(A1) $\sqrt{\frac{\pi}{2}}$; (A2) $\sqrt{2\pi}$; (A3) $\sqrt{\frac{1}{2\pi}}$; (A4) $\sqrt{\frac{2}{\pi}}W$; (A5) $\frac{W}{\pi}$; (A6) $\frac{W}{2\pi}$; (A7) $2\frac{W}{\pi}$;
(A8) $\sqrt{\frac{1}{2\pi}}W$; (A9) $\sqrt{2\pi}W$; (B1) $x\sin(xW)$; (B2) $\operatorname{sinc}\left(\frac{W}{\pi}x\right)$; (B3) $\operatorname{sinc}(\pi Wx)$;
(B4) $\operatorname{sinc}(Wx)$; (B5) $\sin(xW)$; (C1) $\frac{e^{-2t}}{n!}$; (C2) $\frac{e^{2t}}{n!}$; (C3) $\frac{e^{t}}{n!}$; (C4) $\frac{e^{-t}}{2n!}$;
(C5) $(-1)^n\frac{d^n}{dt^n}\left(t^ne^{-t}\right)$; (C6) $\frac{d^n}{dt^n}\left(t^ne^{t}\right)$; (C7) $\frac{d^n}{dt^n}\left(t^ne^{-t}\right)$; (C8) $(-1)^n\frac{d^n}{dt^n}\left(t^ne^{t}\right)$;
(D1) none of above

(12) (7%) Solve the differential equation: $x^3 \dfrac{d^3 y}{dx^3} - 5x^2 \dfrac{d^2 y}{dx^2} + 18x \dfrac{dy}{dx} - 26y = 0$.

(13) (7%) Solve the differential equation: $x \dfrac{dy}{dx} + 2y = xy^3$.

科目：工程數學A(5002)　參考用　校系所組：中大光電科學與工程學系、照明與顯示科技研究所
清大電機工程學系甲組、光電工程研究所
清大電子工程研究所、工程與系統科學系丁組
清大動力機械工程學系乙組
陽明醫學工程研究所醫學電子組、
陽明生醫光電工程研究所理工組B

1. Consider the ODE $(3y^2+x+1)dx+2y(x+1)dy=0$.

(1) (4%) Find an integrating factor for the ODE.

(2) (4%) Given $y(0)=1$, solve the initial value problem.

2. Consider a mass-spring system governed by the ODE $y''+6y'+18y=-90\sin(6t)$.

(1) (3%) How would you describe this system (choose one below)?
(A) Undamped; (B) Underdamped; (C) Critical damped; (D) Overdamped.

(2) (5%) Find the steady-state solution.

3. Consider the ODE $x^3y'''+8x^2y''+9xy'-9y=0$ for $x>0$.

(1) (5%) Find a basis of solutions $\{y_1(x), y_2(x), y_3(x)\}$ for the ODE.

(2) (4%) Given initial conditions $y(1)=0$, $y'(1)=-2$, and $y''(1)=2$, solve the initial value problem.

4. (5%) Bessel function of the first kind of order ν, $J_\nu(x)$, is one solution of the Bessel equation, $x^2y''+xy'+(x^2-\nu^2)y=0$. The general solution of the ODE, $x^2y''+xy'+(4x^4-\frac{1}{9})y=0$, can be expressed as $y(x)=C_1J_\nu\left(ax^2\right)+C_2J_{-\nu}\left(ax^2\right)$. Determine the values of a and ν.

5. (10 %) Use Laplace transform to solve $xy''+(1-x)y'+ky=0$.

(A) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^ke^{-t}]$ (B) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^ke^t]$ (C) $y=\frac{e^t}{k}\frac{d^k}{dt^k}[t^ke^{-t}]$ (D) $y=\frac{e^t}{k!}\frac{d^k}{dt^k}[t^{-k}e^{-t}]$ (E) $y=\frac{e^{-t}}{k!}\frac{d^k}{dt^k}[t^ke^{-t}]$
(F) $y=\frac{e^{-t}}{k}\frac{d^k}{dt^k}[t^ke^{-t}]$ (G) $y=\frac{e^k}{t!}\frac{d^k}{dt^k}[t^ke^{-t}]$ (H) none of the above

6. (10 %) Find the Fourier transform of $f(x)=\sqrt{\frac{\pi}{2}}$ if $|x|<2$ and $f(x)=0$ otherwise.

(A) $f(w)=\frac{\sin w}{w}$ (B) $f(w)=\frac{\sin w}{2w}$ (C) $f(w)=\frac{\cos w}{w}$ (D) $f(w)=\frac{\cos w}{2w}$
(E) $f(w)=\sqrt{\frac{\pi}{2}}\frac{\sin w}{w}$ (F) $f(w)=\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$ (G) $f(w)=\frac{\cos 2w}{w}$ (H) none of the above

注意：背面有試題

7. Consider the problem

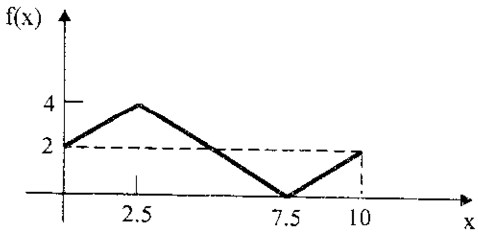
$u_{tt} - 4u_{xx} = 0 \qquad 0 < x < 10$

$u(0,t) = u(10,t) = 2 \qquad 0 < t$

$u(x,0) = f(x) \qquad 0 < x < 10$

$u_t(x, t=0) = 0 \qquad 0 < x < 10$

$f(x)$ is shown in the following figure.



(1) (7%) What is $u(2,1)$ (the value of u at position x = 2 when t = 1)?

(A) 0.8 (B) 1.2 (C) 1.6 (D) 2 (E) 2.4 (F) 2.8 (G) 3.2
(H) none of the above.

(2) (6%) What is the lowest frequency (cycles per time) of the motion of u?

(A) 0.05 (B) 0.1 (C) 0.2 (D) 0.4 (E) 0.8 (F) 1.6 (G) 3.2
(H) none of the above.

8. (7%) The temperature distribution of a thin bar is described by a 1-D heat equation

$u_t - 4u_{xx} = 0 \qquad 0 < x < 10$

The boundary and initial conditions are given as follows:

$u(0,t) = u(10,t) = 0 \qquad 0 < t$

$u(x,0) = \sin\frac{\pi x}{10} \qquad 0 < x < 10$

The peak temperature is located at the position x = 5 at all time. At what time will the peak temperature reduce to 1/e of its initial value?

(A) $\pi/10$ (B) $\pi^2/100$ (C) $10/\pi$ (D) $100/\pi^2$ (E) $\pi/5$
(F) $\pi^2/25$ (G) $5/\pi$ (H) none of the above.

9. (20%) Evaluate the principal value of the integral $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^3 + x^2 + 3x - 5}\, dx$.

10. (10%) Find the eigenvalues and corresponding normalized eigenvectors (norm equals to 1) for the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 6 & 4 & 2 \end{bmatrix}$. What are those for the transpose matrix A^T?

清華大學

微機電系統工程研究所

91~97 學年度
工程數學考古題

九十學年度 微機電系統工程研究 所 ___ 組碩士班研究生招生考試
科目 應用數學 科號 2201 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. Solve the differential equations
   (a) $(2x^2+y)dx+(x+2x^2y-x^4y^3)dy=0$ (10%)
   (b) $y''(x)+y(x)=\sin x+xe^x$ (15%)

2. (a) Let V be the vector space of 2 x 2 matrices over **R**. Determine whether the matrices **A**, **B**, **C** $\in$ V are dependent where:

$$\mathbf{A}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$ (5%)

   (b) Let **A** be an n x n matrix. Show that for all n x 1 vector **x**, $|\mathbf{x}^T\mathbf{A}\mathbf{x}|\le\|\mathbf{A}\|\|\mathbf{x}\|^2$, where $\|\bullet\|$ denotes a norm. (5%)

3. The matrix **A** is a 2 x 2 constant matrix with a pair of complex conjugate eigenvelues $\alpha+j\beta$ and $\alpha-j\beta$. Find the transformation matrix **P** such that

$$\mathbf{B}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}$$ (15%)

4. (a) Prove that the Fourier transform of the convolution product of f(t) and g(t) is given by

$$\mathcal{F}[f(t)*g(t))]=\sqrt{2\pi}\mathcal{F}[f(t)]\mathcal{F}[g(t)]$$ (13%)

   (b) Determine the Fourier transform of the function

$$f(t)=\frac{5e^{3it}}{t^2-4t+13}$$ (12%)

5. Write the solutions of the following boundary value problems

$$\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2}\quad (0<x<L,t>0)$$
$$u(0,t)=u(L,t)=0\quad (t>0)$$
$$u(x,0)=L[1-\cos(\frac{2\pi x}{L})]\quad (0<x<L)$$ (25%)

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九十二學年度 微機電系統工程研究 (系)所 ____ 組碩士班研究生招生考試

科目 應用數學 科號 2202 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Solve the following first-order differential equation:

$$y'(x)=\frac{2y}{x(y-1)} \qquad (10\%)$$

2. Solve the following second-order differential equation:

$$(x+1)^2 y''(x)+2(x+1)y'(x)-4y(x)=2x+1 \qquad (15\%)$$

3. Find the eigenvalues and the corresponding eigenvectors for the matrix A.

$$A=\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 3\\ 0 & 0 & 2\end{bmatrix} \qquad (15\%)$$

4. (a) Prove $\mathcal{L}[\dot{f}(t)]=sF(s)-f(0)$ if the Laplace transform of $f(t)$ is $\mathcal{L}[f(t)]=F(s)$. (5%)

   (b) Find the inverse Laplace transform, $\mathcal{L}^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$ (5%)

5. If $\overline{V}$ is a vector function, show the following

   (1) $\nabla\bullet(\nabla\times\overline{V})=0$. (5%)

   (2) $(\overline{V}\bullet\nabla)\overline{V}=(\nabla\times\overline{V})\times\overline{V}+\nabla(V^2/2)$. (5%)

   (3) $(\nabla\times\overline{V})\times\overline{V}$ is normal to $\overline{V}$. (5%)

6. Evaluate the integrals $\int_{-\infty}^{\infty}\frac{\cos kx}{(x-a)^2+b^2}dx$ and $\int_{-\infty}^{\infty}\frac{\sin kx}{(x-a)^2+b^2}dx$ for k>0 by using Fourier Transform.

   (Hint: $e^{ikx}=\cos kx+i\sin kx$) (15%)

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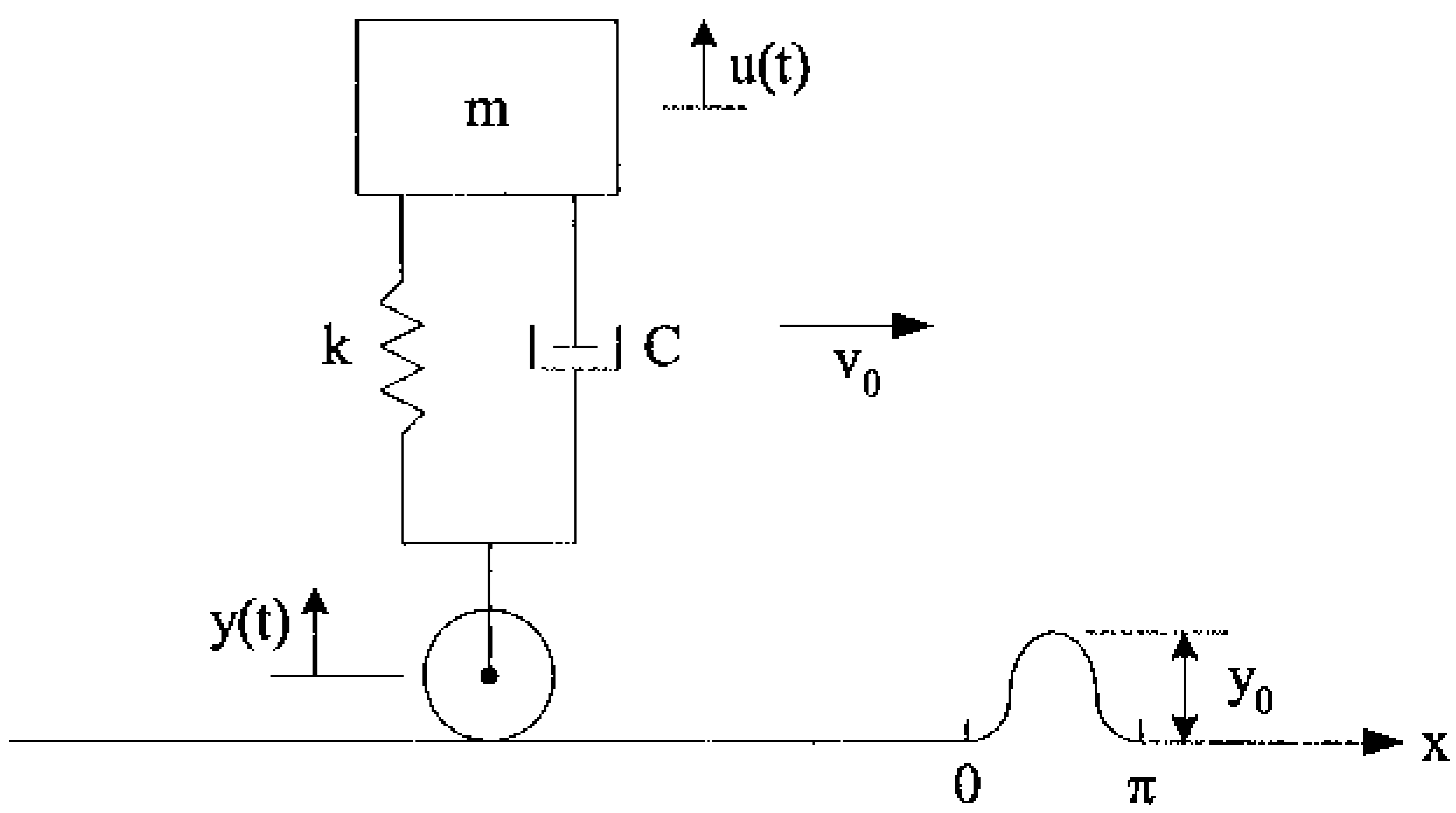
九十二學年度 微機電系統工程研究 (系)所 ＿＿ 組碩士班研究生招生考試

科目 應用數學 科號 2202 共 2 頁第 2 頁 *請在試卷【答案卷】內作答

7. A monocycle shown below moves at a constant velocity v_0 hitting a bump along x direction. Assume the mass of the suspension and wheel assemble is negligible.



(a) Please derive the second order govern equation of this system as below

$$\frac{d^2u(t)}{dt^2}+\frac{d[u(t)]}{dt}+5u(t)=y(t) \qquad where \begin{cases} y=y_0\sin^2(8t) & (0<x<\pi/8) \\ =0 & (x<0, x>\pi/8) \end{cases}$$

- The relation between spring constant k and mass m is $k/m=5$, the damping constant C and mass m is $C/2m=1$, and the constant velocity is $v_0=8$.
- The bump condition: $\begin{cases} y=y_0\sin^2 x & (0<x<\pi) \\ =0 & (x<0, x>\pi) \end{cases}$.

(Hint: Start from relative parameter u(t)-y(t). Find relationship between x and t, then make derived PDE to be u(t) only equation.) (3%)

(b) Solve the PDE you derived above if $C=0$ (no damping case). (14%)

(c) Continue from (b), and find $u(t)/y_0$ if initial conditions are $u(0)=0$ and $\left.du(t)/dt\right|_{t=0}=0$. (3%)

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九十三學年度 微機電系統工程研究 (系)所 甲 組碩士班入學考試

科目 數學 科號 2402 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

1. Let $f = 4x^2 + xy^2 + 9y^3z^2$ (scalar function) and $\boldsymbol{v} = xz\boldsymbol{i} + (x-y)^2\boldsymbol{j} + 2x^2yz\boldsymbol{k}$ (vector function). Find
   (a) $\nabla^2 f$ (5%)
   (b) $curl(grad\ f)$ (5%)
   (c) $\nabla f \bullet curl\boldsymbol{v}$ (5%)

2. Solve the following first-order differential equation for $u(t)$:

   $\frac{du}{dt} = \exp(t+u)$, $u(0) = 1$ (10%)

3. For a matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$,
   (a) Find the 2x2 matrix ***P*** and ***D***, such as ***P***$^{-1}$***AP***=***D*** where D is a diagonal matrix. (5%)
   (b) Find the eigenvalues and the corresponding eigenvectors for $f(A)$, where $f(x) = 5x + 2$. (5%)

4. Use Laplace transforms to solver the following equations for y(t)
   (a) $\frac{d^2y}{dt^2} + y = \cos(2t)$, where $t \geq 0$, $y(0) = 1$, and $\frac{dy(0)}{dt} = 0$. (10%)

   (b) $y(t) = 6t + \int_0^t y(t-s)\sin(s)ds$, $t \geq 0$. (10%)

5. Evaluate the following integrals by using Fourier Transform.

   (a) $\int_0^\infty \frac{dx}{x^2+1}$ (5%)

   (b) $\int_0^\infty \frac{\sin(ax)}{\sinh(bx)}dx$ (10%)

   (Hint: $e^{ikx} = \cos kx + i\sin kx$ and $\sinh(bx) = \frac{\exp(bx) - \exp(-bx)}{2}$)

6. Solve the following second-order differential equation for $u(t)$:

   $t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + 4u = \sin[\ln(t)]$ (10%)

7. Below is so called the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) Find the deflection $u(x,t)$ of the vibrating string based on the following conditions, (14%)

- Boundary conditions: $u(0,t) = 0$ and $u(L,t) = 0$ for all t.
- Initial conditions: $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}\Big|_{t=0} = g(x)$.

(b) Find the deflection $u(x,t)$ by using the derived result from (a) and the following functions and parameters, (6%)

- $c = 1$ and $L = \pi$.
- $f(x) = 0$ and $g(x) = 0.1\sin(2x)$.

1. (a) Solve the first-order differential equations

$$x^2(y+1)dx + y^2(x-1)dy = 0 \qquad (10\%)$$

(b) Solve the differential equations where λ is real.

$$y''(x) + \lambda y = 0,\ y(0) = y'(\pi) = 0,\ 0 \le x \le \pi \qquad (10\%)$$

(c) Solve the simultaneous differential equations

$$\begin{aligned} \dot{y}_1 - 3y_1 &= y_2 \\ \dot{y}_2 - y_2 &= -y_1 \end{aligned} \qquad (15\%)$$

2. Give the periodic function

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

Find the Laplace transform of $[f(t)]$. (10%)

3. (a) Let $p(s)$ is a polyminal in s. Show that if λ is an eigenvalue of a square matrix A with eigenvector $\boldsymbol{x}$, then $p(\lambda)$ is an eigenvalue of $p(A)$ with the same eigenvector $\boldsymbol{x}$. (10%)

(b) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^k for any integer k. (15%)

4. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - 3y^2 z\mathbf{k}$, S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5, and $\boldsymbol{n}$ is the unit vector of the surface S. (15%)

5. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$, where a is constant. (7%)

(b) Use the result of (a) to evaluate $\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha$. (8%)

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九十五學年度 微機電系統工程研究 (系)所 組碩士班入學考試

科目 工程數學 科號 2403 共 1 頁第 1 頁 *請在試卷【答案卷】內作答

1. Consider the following equation for the temperature u(x) in a chemical reacting slab of material:

$$\frac{d^2u}{dx^2}+\lambda\left[e^u-1\right]=0,\ \ 0<x<1 \ \text{ and } \ u(0)=u(1)=0.$$

Find $u_0(x)$ and λ_0 for a small amplitude positive solution of the form

$$u(x)=\varepsilon\, u_0(x)+\varepsilon^2\, u_1(x)+(\text{higher order terms of } \varepsilon) \ \text{ and } \ \varepsilon=\lambda-\lambda_0$$ (20%)

2. Use Laplace transform to solve the y(t):

$$\frac{d^2y}{dt^2}+9y=f(t),\ \ y(0)=\frac{dy}{dt}(0)=0,$$

$f(t)=t$ if $0<t<a$ and $f(t)=0$ else. Here a is a positive number (15%)

3. Solve the equation

$$\frac{d^2y}{dt^2}-y=e^t,\ \ y(0)=\frac{dy}{dt}(0)=0$$ (10%)

4. AX=Y+ **noise**, where $A=\begin{bmatrix}1&-2&-1&2&1\\1&-2&0&3&0\\-1&2&2&-1&-2\\2&-4&-3&3&3\\-1&2&-1&-4&1\end{bmatrix}$, $Y=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$

(a) Find the Rank of A (10%)

(b) Find the complete base for the row space of A (10%)

(c) Find a general solution for vector X based on the row space that you got (10%)

5. $\vec{z}_k=A\vec{z}_{k-1}$ where $\vec{z}_k=\begin{bmatrix}x_{k+1}\\x_k\end{bmatrix}$, $A=\begin{bmatrix}a&b\\1&0\end{bmatrix}$ and $k=1,2,3,\ldots$

(a) Derive an expression for $\vec{z}_k$ in terms of $\vec{z}_0$ (10%)

(b) The ratio $\frac{x_{k+1}}{x_k}$ approach a constant when k becomes very large. For the case in which

$a=-1$ *and* $b=1$, determine the ration $\frac{x_{k+1}}{x_k}$. Hint: It is possible to answer this question without a lot

of computation. (15%)

1. Laplace Transform can be used to solve differential equations.
The model of the system in the figure 1 is:

$$m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$$
$$m_2 y_2'' = -k_2 (y_2 - y_1) - k_3 y_2$$

while

$$m_1 = m_2 = 10 \text{ kg},$$
$$k_1 = k_3 = 20 \text{ kg/sec}^2$$
$$k_2 = 40 \text{ kg/sec}^2$$

(a) Please find the solutions [$y_1(t)$ = ?, $y_2(t)$ = ?] which satisfying the initial conditions:

$$y_1(0) = y_2(0) = 0$$
$$y_1'(0) = 1 \text{ m/sec}$$
$$y_2'(0) = -1 \text{ m/sec}$$

through Laplace Transform (15 points)

(b) When

$$y_1(0) = y_2(0) = 1 \text{ meter}$$
$$y_1'(0) = y_2'(0) = 0$$

Please find the solutions [$y_1(t)$ = ?, $y_2(t)$ = ?] through Laplace Transform and compare the solutions in (a) and (b) (frequency, type of motion…etc) (12 points)

(c) These differential equations are also a typical eigenvalue problem. Please solve the (a) by the method of eigenvalue problem. (15 points)

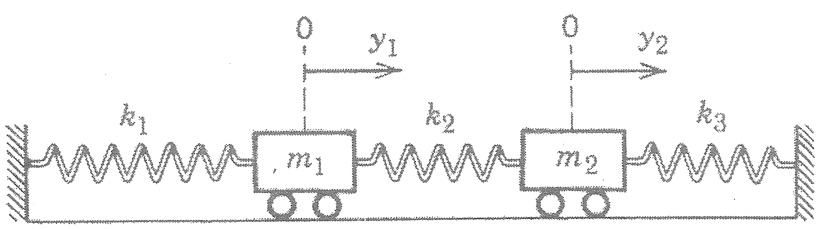

Figure 1. Mass-Spring System

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97 學年度奈米工程與微系統研究所系（所）＿＿＿＿組碩士班入學考試

科目 工程數學 科目代碼 1803 共 2 頁第 2 頁 *請在【答案卷卡】內作答

2. Partial Differential Equations

Find the solution u(x,y) of following equations using separation variables:

(a) $u_x = y\,u_y$ (7 points)

(b) $a\,y\,u_x = b\,x\,u_y$ (8 points)

(c) $x^2 u_{xy} + 3y^2 u = 0$ (8 points)

3. Find the solution of the following Bernoulli equation. (15 points)

$$y' - 4y = 4y^2$$

4. (*Vector transformation in 3D space*) A vector in 3D space can be expressed by different coordination, for example, in both rectangular and cylindrical systems,

$\bar{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$, where $[A_\rho, A_\phi, A_z]^T = Q_{rc}[A_x, A_y, A_z]^T$.

(a) Find the coordinate transformation matrix Q_{rc} (15 points.)

(b) Similarly, $[A_x, A_y, A_z]^T = Q_{cr}[A_\rho, A_\phi, A_z]^T$. Find Q_{cr} (5 points)