成功大學

土木工程學系

91~97 學年度
工程數學考古題

1. Given a homogeneous solution of the following differential equation

$$x^2y'' + xy' + (x^2 - 1/4)y = x^{3/2}, \qquad (15)$$

as $y_1 = \sin x/\sqrt{x}$, find the particular solution.

2. Find the solution of the differential equation $y'' + cy' + y = r(t)$, with $c > 0$ and $r(t)$ given as

$$r(t) = \frac{t}{12}(\pi^2 - t^2) \quad \text{if} \quad -\pi < t < \pi \quad \text{and} \quad r(t + 2\pi) = r(t), \qquad (20)$$

3. Given the eigenvalues of a matrix

$$A = \begin{pmatrix} 4 & 3 & 9 & 9 \\ -8 & 3 & 5 & -4 \\ -8 & 0 & -2 & -8 \\ -16 & 6 & 14 & -5 \end{pmatrix}, \qquad (15)$$

as $\lambda_1 = -0.2776 + 18.6896i$, $\lambda_2 = -0.2776 - 18.6896i$, $\lambda_3 = -0.1042$, and $\lambda_4 = 0.6593$, where $i = \sqrt{-1}$.
Find the eigenvalues of A^{-1}.

4. Calculate the work done by a force

$$\vec{F} = x^2\vec{i} - xy\vec{j}, \qquad (15)$$

fron point (1,0) to (-1,0) along a curve of $x^2 + y^2/4 = 1$ in the upper plane (i.e. $y \geq 0$).

5. Given a velocity field as

$$\vec{v} = 7x\vec{i} - z\vec{k}$$

find the surface integral

$$I = \int\int_S \vec{v} \bullet \vec{n} dA, \qquad (15)$$

where $\vec{n}$ is a unit outer normal vector for a sphere $x^2 + y^2 + z^2 = 4$

6. Solve the following partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 1, \qquad (20)$$

with boundary conditions $\phi(0, y) = \phi(a, y) = \phi(x, 0) = \phi(x, b) = 0$.

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1. (20%) Solve the boundary value problem

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(\mathbf{x})=-2,\ \mathbf{x}\in\Omega;\quad \Phi(\mathbf{x})=0,\ \mathbf{x}\in\partial\Omega$$

where Ω is a circle centered at the point $\mathbf{x}_0$ with radius R, i.e.

$$\Omega:|\mathbf{x}-\mathbf{x}_0|\le R,\quad \partial\Omega:|\mathbf{x}-\mathbf{x}_0|=R.$$

Express $\Phi(\mathbf{x})$ in terms of $\mathbf{x},\ \mathbf{x}_0$ and R.

2. (i) (10%) Evaluate

$$\oint_C \frac{z^2}{2z-1}dz,\ C:|z|=1.$$

(ii) (10%) Given $z_1=1+i$, $z_2=1+i\sqrt{3}$, $z_3=\sqrt{3}-i$. Find

$$\arg\left(\frac{z_1z_2}{z_3}\right).$$

3. (i) (10%) Setting

$$x=e^z,$$

show that

$$x\frac{d}{dx}=\frac{d}{dz},\quad x^2\frac{d^2}{dx^2}=\frac{d^2}{dz^2}-\frac{d}{dz}.$$

(ii) (10%) Solve the differential equation

$$x\frac{dy}{dx}+y+x^2y^2=0.$$

Hint: introduce a new variable $v=y^{-1}$.

4. (i) (10%) Solve the algebraic set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1 \\ 3x_1 - 3x_2 - 5x_3 &= 1. \end{aligned}$$

(ii) (10%) Suppose the matrices A and B possess the same set of linearly independent eigenvectors so that they can be diagonalized through

$$S^{-1}AS = \Lambda_A, \quad S^{-1}BS = \Lambda_B,$$

where Λ_A and Λ_B are diagonal matrices. Show that in this case $AB = BA$.

5. (i) (10%) Evaluate the line integral

$$\int_C \mathbf{f} \cdot \mathbf{dx}$$

where $\mathbf{f}(\mathbf{x}) = 5z\,\mathbf{i} + xy\,\mathbf{j} + x^2z\,\mathbf{k}$ and C is the straight line from $(0,0,0)$ to $(1,1,1)$.
(ii) (10%) Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}d\sigma$$

where $\mathbf{F} = 2x\,\mathbf{i} + 3y\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the hemisphere $x^2+y^2+z^2 = 9$, $z \geq 0$.

1. (i) (10%) Expand $f(z)$

$$f(z)=\frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for $1<|z|<3$.
(ii) (10%) Evaluate the integral

$$\oint_C \bar{z}dz$$

where C is the circle $|z|=2$.

2. (i) (10%) For a curve $x=t^2+1,\ y=4t-3,\ z=2t^2-6t$, determine the unit tangent vector at the point where $t=2$.
(ii) (10%) Evaluate

$$\iint_S \mathbf{x}\cdot\mathbf{n}dS$$

where $\mathbf{x}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\mathbf{n}$ is the outward unit normal to S, and S is the surface of the sphere

$$(x-1)^2+(y+3)^2+z^2=4.$$

3. (10%) (i) Given the 3×3 matrix $\mathbf{P}$ of the form

$$\mathbf{P}=\begin{pmatrix} 7 & -2 & -4 \\ 3 & 0 & -2 \\ 6 & -2 & -3 \end{pmatrix}.$$

Find a matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{P}\mathbf{C}$ becomes a diagonal matrix.
(ii) (10%) Given a 3×3 symmetric matrix $\mathbf{A}$ defined by

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

what are the constraint conditions for the elements a_{ij} to ensure that the matrix $\mathbf{A}$ is positive definite?

(背面仍有題目,請繼續作答)

4. (20%) Determine the steady-state temperature $T(r,\theta)$ at points of the sector $0 \le \theta \le \alpha$, $0 \le r \le a$ of a circular plate if the temperature is maintained at zero along the straight edges and at a prescribed distribution $T(a,\theta) = T_0$ = constant when $0 < \theta < \alpha$, along the curved edge. (Hint: solve $\nabla^2 T$ by separation of variables).

5. (i) (10%) Solve

$$\frac{dy}{dt} + 4y = 3H(t-2)\,e^{-t};\; y(0) = 0,$$

where $H(x)$ is the unit step function.
(ii) (10%) Solve

$$x^2y'' - 3xy' + 3y = \ln x;\; y(1) = 1,\; y'(1) = 2.$$

1. (i) (10%)

$$\text{(A) } \mathbf{F} = -\nabla\varphi, \quad \text{(B) } \oint \mathbf{F}\cdot d\mathbf{r} = 0, \quad \text{(C) } \nabla \times \mathbf{F} = 0.$$

In the following choices which one (a,b,c,d,e,f) is correct? Explain why.

$$\begin{aligned}
&\text{(a) } (A) \Leftrightarrow (B) \Leftrightarrow (C),\\
&\text{(b) } (A) \Leftrightarrow (B) \not\Leftrightarrow (C),\\
&\text{(c) } (A) \Leftrightarrow (C) \not\Leftrightarrow (B),\\
&\text{(d) } (A) \not\Leftrightarrow (B) \Leftrightarrow (C),\\
&\text{(e) } (A) \not\Leftrightarrow (B) \not\Leftrightarrow (C).
\end{aligned}$$

(ii) (10%) Evaluate the integral

$$\oint_C (y - \sin x)\, dx + \cos x \; dy,$$

where C is the triangle from $(0,0) \to (\pi/2, 0) \to (\pi/2, 1) \to (0,0)$.

2. (i) (10%)Given two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, suppose that there exists an invertible $n \times n$ matrix $\mathbf{C}$ such that

$$\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}.$$

Show that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvelues.

(ii) (10%) Given the quadrative form

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = d,$$

in which $\mathbf{A}$ is an $n \times n$ symmetric matrix, $\mathbf{x}$ is an $n \times 1$ matrix and d is a scalar. What are the conditions for the matrix $\mathbf{A}$ so that the value d is always positive for any given $\mathbf{x}$?

(背面仍有題目,請繼續作答)

3. (i) (10%) Find the solution of

$$\int_0^{2\pi} \frac{dx}{1 - 2p\cos x + p^2}, \quad 0 < p < 1.$$

(ii) (10%) Find the image of the circle $|z| = 2$ after the mapping

$$w = z + \frac{1}{z},$$

where $z = x + iy$ and $w = u + iv$.

4. (i) (10%) Setting

$$\varphi(x.y) = e^{-(ax+by)} U(x.y),$$

transform the partial differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + 2a\frac{\partial \varphi}{\partial x} + 2b\frac{\partial \varphi}{\partial y} = 0$$

into a different partial differential equation with unknown $U(x.y)$.
(ii) (10%) Find the general solution of $z(x, y)$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

5. (i) (15%) Solve the differential equation by the method of Laplace transform

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + 2y = t, \quad y(0) = \frac{dy(0)}{dt} = 1.$$

(ii) (5%) What are Legendre polynomials?

1. (i) (10%) What is Bessel's equation of order n? Write down the solutions for $n =$ integer and $n \neq$ integer. (ii) (10%) What is Legendre's equation? Describe what you know about Legendre polynomials.

2. (20%) We consider the relation

$$x = r\cos\theta,\ y = r\sin\theta.$$

Then

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}},\ \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Obviously, we can see that

$$\frac{\partial x}{\partial r} = \cos\theta. \tag{1}$$

(i) derive

$$\frac{\partial r}{\partial x}$$

in terms of r and θ, and compare with Eq. (1), (ii) show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

3. (20%) Let Ω be a simply connected region in the xy-plane bounded by a piecewise smooth curve $\partial\Omega$. Let $\mathbf{T}$ denote the unit tangent vector to $\partial\Omega$ and $\mathbf{n}$ be the unit normal vector to $\partial\Omega$. Given a vector $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ in the plane, (i) describe the physical (or mathematical) meanings of the two line integrals

$$\oint_{\partial\Omega} \mathbf{F}\cdot\mathbf{T}ds, \quad \text{and} \quad \oint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}ds,$$

where s denotes the parameter of arc length, (ii) transform these two line integrals into double integrals in the plane.

(背面仍有題目,請繼續作答)

4. (i) (10%) Find the solution of

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, \quad \text{for } a > |b|.$$

(ii) (10%) Given $z = x + iy$, evaluate the integral

$$\int_C y\,dz$$

where C is the straight line joining $z = 1$ to $z = i$.

5. Given the second-order linear partial differential equation of two independent variables with constant coefficients

$$a\frac{\partial^2 w}{\partial x^2} + 2b\frac{\partial^2 w}{\partial x \partial y} + c\frac{\partial^2 w}{\partial y^2} + nw = f(x,y), \tag{2}$$

where a, b, c and n are constants. Using the substitution

$$\begin{aligned} u &= x\cos\alpha + y\sin\alpha, \\ v &= -x\sin\alpha + y\cos\alpha, \end{aligned}$$

and transform Eq. (2) into the form

$$A\frac{\partial^2 w}{\partial u^2} + 2B\frac{\partial^2 w}{\partial u \partial v} + C\frac{\partial^2 w}{\partial v^2} + nw = f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha).$$

(i) (15%) Find the expressions of A, B and C in terms of a, b, c and α, (ii) (5%) Under what condition, the partial differential equation (2) is referred to as an elliptic type.

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1. Determine the nature of the singularity (if any) at $z=0$ for the following $f(z)$. Can you expand these functions in powers of z convergent in a punctured disk $0<|z|<R$. (25%)

   (a) $\sin(1/z)$
   (b) $(\sin z)/z$
   (c) $(\sin z)/z^2$
   (d) $1/\sin(1/z)$
   (e) $z\sin(1/z)$

2. Are the following statements true or false? If it is false, explain the reason. (16%)

   (a) If $u(x,y)$ is harmonic in D, then it is the real part of an analytic function $f(z)$ in D.
   (b) The real and imaginary parts of a complex analytic function are harmonic.
   (c) If two analytic functions have the same real part $u(x,y)$, then $f(z)=g(z)$ identically.
   (d) If $f(z)=u(x,y)+iv(x,y)$ with $u(x,y), v(x,y)$ harmonic, then $f(z)$ is analytic.

3. Solve the following equation (15%)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=0$$

   in a unit disk with $u=1+\theta$ on the boundary.

4. Let $\boldsymbol{F}(x,y,z)=(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})/r^n$, where $r=\sqrt{x^2+y^2+z^2}$ and n is a positive integer.

   (a) Show that $div\boldsymbol{F}=(3-n)/r^n$ (4%)

   (b) Evaluate the surface integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dS$ for $n=2$ where S is the

（背面仍有題目,請繼續作答）

sphere $x^2 + y^2 + z^2 = a^2$. Can you use the divergence theorem?(5%)

(c) Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$ for $n = 3$ where S is the sphere $x^2 + y^2 + z^2 = a^2$. Can you use the divergence theorem? (5%)

5. Determine all possible solutions for the following equation

$$\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where λ is any real number.(15%)

6. Bessel's equation is:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

Determine the nature of the singularity at $x = \infty$ by transforming the independent variable to $z = 1/x$. (15%)

1. Consider the second-order homogeneous linear differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

a) Find the two linearly independent solutions f_1 and f_2 of this equation which are such that

$$f_1(0) = 1 \text{ and } f_1'(0) = 0$$

and

$$f_2(0) = 0 \quad \text{and} \quad f_2'(0) = 1 \quad (5\%)$$

b) Express the solution

$$3e^x + 2e^{2x}$$

as a linear combination of the two linearly independent solutions f_1 and f_2 defined in (a). (5%)

2. Consider the differential equation

$$(4x + 3y^2)dx + 2xydy = 0$$

a) Show that this equation is not exact.(5%)

b) Find an integrating factor of the form x^n, where n is a positive integer.(5%)

c) Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation. (5%)

3. The function f has at (1,-1) a directional derivative equal to $\sqrt{2}$ in the direction toward (3,1), and $\sqrt{10}$ in the direction toward (0,2).

（背面仍有題目,請繼續作答）

a) Find the value of $\partial f/\partial x$ and $\partial f/\partial y$ at (1,-1).(5%)

b) Determine the derivative of f at (1,-1) in the direction toward (2,3). (5%)

4. Find a unit tangent vector to the curve of intersection of the plane $y-z+2=0$ and the cylinder $x^2+y^2=4$ at the point (0,2,4) (10%)

5. Evaluate the line integral

$$\int_c \frac{-ydx+(x-1)dy}{(x-1)^2+y^2}$$

where c is any piecewise smooth simple closed curve containing the point (1,0) in its interior. (15%)

6. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)}dx$$

by complex variable methods. (15%)

7. Show that any function f(t) can be expressed as the sum of two component functions, one of which is even and the other odd. (10%)

8. An important property of the Laplace transform is the convolution theorem. State this theorem and prove it. (15%)

1. Find the particular solution of the following equation

$$(x+1)\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=e^x(x+1)^2. \qquad (20)$$

2. Solve the following differential equation

$$\frac{d\mathbf{x}}{dt}=\begin{pmatrix}5 & 8\\ -6 & -9\end{pmatrix}\mathbf{x}+\begin{pmatrix}1\\ t\end{pmatrix}$$

where

$$\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \qquad (20)$$

3. The temperature distribution in a homogeneous spherical solid filling the closed region $x^2+y^2+z^2\le 1$ at time t is given by $u=(z^2-z)e^{-2t}$. Let $\vec{n}$ be the unit outer normal on the boundary of the sphere. Find the point at which $\partial u/\partial n$ is minimum. (15)

4. Given an analytic function $f(z)=F_1(x,y)+iF_2(x,y)$, where $z=x+iy$ and $i=\sqrt{-1}$. If the real part $F_1(x,y)$ and the imaginary part $F_2(x,y)$ of $f(z)$ serve as the components of a vector $\vec{F}$, i.e.

$$\vec{F}=F_1\vec{i}+F_2\vec{j}, \qquad (10)$$

where $\vec{i}$ and $\vec{j}$ denote the unit vector in x- and y-direction respectively. Then, is the vector $\vec{F}$ a conservative one ? why ?

5. Given a velocity field as

$$\vec{v}=y\vec{i}-z\vec{j}+yz\vec{k}$$

find the surface integral

$$I=\int\int_S \vec{v}\bullet\vec{n}\,dA, \qquad (15)$$

where $\vec{n}$ is a unit normal vector in the outer direction of the surface

$$S: x=\sqrt{y^2+z^2}; \qquad y^2+z^2\le 1$$

6. The displacement $u(x,t)$ of a semi-infinite string is governed by the following partial differential equation

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}, \quad 0<x, \quad 0<t, \qquad (20)$$

where c is a constant. With the initial conditions

$$u(x,0)=0, \qquad \frac{\partial u(x,0)}{\partial t}=0,$$

and the excitation $f(t)$ at one end of string, that is,

$$u(0,t)=f(t)$$

then, what is the solution of $u(x,t)$?

1. Solve the following differential equation

$$(x+1)^2\frac{d^2y}{dx^2}-3(x+1)\frac{dy}{dx}+2y=0 \qquad (20)$$

with boundary conditions $y(0)=0$ and $\frac{dy(1)}{dx}=3$

2. Use Laplace transform to solve the deflection u(x) of a fixed-end beam of length l subjected to a concentrated loading P as shown in the following differential equation

$$EI\frac{d^4u}{dx^4}=P\delta(x-\frac{l}{3}), \qquad 0\le x\le l,$$

with the boundary conditions $u(0)=u(l)=0$ and $\frac{du(0)}{dx}=\frac{du(l)}{dx}=0$,

where $\delta(.)$ is the Dirac delta function and the rigidity EI and loding P are constant. (20)

3. Prove that

(a) The eigenvalues of a Hermitian matrix are always real. (10)

(b) The eigenvalues of similar matrices are the same. (10)

4. Calculate the following surface integral

$$I_s=\int_S \vec{F}\bullet\vec{n}dS,$$

where the vector field $\vec{F}=2z\vec{i}+(x-y-z)\vec{k}$,

$\vec{n}$ denotes the unit outer normal vector of the surface S: $z=x^2+y^2$; $x^2+y^2\le 6$, (15)

5. For a wave equation

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial x^2}, \quad 0\le t, \quad 0\le x$$

(a) Show the D'Alembert's solution of the above equation (10)

(b) Solve $\phi(x,t)$ if $\phi(x,0)=\frac{d\phi(x,0)}{dt}=0$ and $\phi(0,t)=[u(t)-u(t-2)](-t^2+2t)$

where $u(.)$ denotes the unit step function. (15)

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1. Given a homogeneous solution of the following differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\frac{1}{4})y=0 \qquad (20)$$

as $y_1(x)=\dfrac{\sin x}{\sqrt{x}}$, find the other solution.

2. Given the differential equations as follows

$$(1-x^2)\frac{d^2y_1}{dx^2}-2x\frac{dy_1}{dx}+ay_1=0 \text{ and}$$

$$(1-x^2)\frac{d^2y_2}{dx^2}-2x\frac{dy_2}{dx}+by_2=0 \qquad (20)$$

where a and b are constants and $a \neq b$.

Is $\int_{-1}^{1} y_1(x)y_2(x)dx=0$ always true? Why?

3. Define the multiplication of matrices as follows

$$Q=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$ where x_1, x_2 and x_3 are any arbitrary real numbers.

Is $Q>0$ always true ? why ? (20)

4. Verify the Green's theorem for the given vector $\vec{F}=3e^x y\vec{j}$ along a triangle contour C with vertices at (1,1), (2,3) and (1,6) (20)

5. Solve the following wave equation for a string of length L

$\dfrac{\partial^2 y}{\partial x^2}=\dfrac{\partial^2 y}{\partial t^2}+\delta(x-a)e^{-i\omega t}$, with the boundary conditions $y(0,t)=y(L,t)=0$,

where $\delta(.)$ is the Dirac's delta function, $0<a<L$ and ω is a constant. (20)

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1. The Bessel function of the first kind is as follows

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}$$, where $\Gamma(.)$ is the gamma function.

Prove

(a) $\dfrac{d(x^{-\nu}J_\nu(x))}{dx} = -x^{-\nu}J_\nu(x)$, (10)

(b) $J_{-n}(x) = (-1)^n J_n(x)$, if n is an integer (10)

2. Solve the following integral equation (20)

$$f(x) = \sin 2x - 2\int_0^x (x-u)^2 f(u)du$$

3. Given a matrix as follows (20)

$$A = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 7 & -3 \\ 0 & -3 & 3 \end{bmatrix},$$ and define $A^2 = AA$, $A^3 = AAA$ ...... and so on,

please calculate the result of A^{50} .

4. A vector field is (20)

$$\vec{V} = y\vec{i} + x\vec{j} + x^2\vec{k} \ ,$$

and the surface is described as

S: $z = 1-(x^2+y^2)$, $0 \le z$,

calculate the following flux integral

$$I = \iint_S \vec{V} \bullet \vec{n} dA$$

where $\vec{n}$ is an outer unit normal vector on the surface.

5. Solve the following partial differential equation (20)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \sin y$$, with the following conditions

$$\phi(0,y) = \phi(1,y) = 0,\ \phi(x,0) = \frac{\partial\phi(x,\pi/2)}{\partial y} = 0$$

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1. Solve the differential equation $\dfrac{dy}{dx}=\dfrac{2x+y}{2x+y+1}$.[Hint: let $u=2x+y$] (20)

2. (a) Explain Cauchy-Riemann equations.
   (b) Give the real part $u(x,y)=x^2-y^2$ of an analytic complex function $f(z)=u(x,y)+iv(x,y)$, find the imaginary part $v(x,y)$.
   (c) Determine the derivative of $f(z)$. (20)

3. (a) Explain half-range Fourier series expansion.
   (b) Expand the function $f(x)=x^2,\quad 0<x<\pi$ in a Fourier series and in a Fourier sine series (half-range expansion). (20)

4. (a) Explain the directional derivative of a function.
   (b) Find the directional derivative of the function $f(x,y)=x+y^2$ at point $(3,4)$ in the direction $2\mathbf{i}+\mathbf{j}$.
   (c) Find the maximum directional derivative of the function $f(x,y)=x+y^2$ at point $(3,4)$.(20)

5. Calculate the double integration $\iint_R xydxdy=?$, where $R:\begin{cases}0<x+y<2\\0<x-y<2\end{cases}$.[Hint: let $u=x+y$, $v=x-y$] (20)

1. Consider the second-order homogeneous linear differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

a) Find the two linearly independent solutions f_1 and f_2 of this equation which are such that

$f_1(0) = 1$ and $f_1'(0) = 0$

and

$f_2(0) = 0$ and $f_2'(0) = 1$ (5%)

b) Express the solution

$3e^x + 2e^{2x}$

as a linear combination of the two linearly independent solutions f_1 and f_2 defined in (a). (5%)

2. Consider the differential equation

$$(4x + 3y^2)dx + 2xydy = 0$$

a) Show that this equation is not exact.(5%)

b) Find an integrating factor of the form x^n, where n is a positive integer.(5%)

c) Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation. (5%)

3. The function f has at (1,-1) a directional derivative equal to $\sqrt{2}$ in the direction toward (3,1), and $\sqrt{10}$ in the direction toward (0,2).

（背面仍有題目,請繼續作答）

a) Find the value of $\partial f/\partial x$ and $\partial f/\partial y$ at (1,-1).(5%)

b) Determine the derivative of f at (1,-1) in the direction toward (2,3). (5%)

4. Find a unit tangent vector to the curve of intersection of the plane $y-z+2=0$ and the cylinder $x^2+y^2=4$ at the point (0,2,4) (10%)

5. Evaluate the line integral

$$\oint_c \frac{-ydx+(x-1)dy}{(x-1)^2+y^2}$$

where c is any piecewise smooth simple closed curve containing the point (1,0) in its interior. (15%)

6. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)}dx$$

by complex variable methods. (15%)

7. Show that any function f(t) can be expressed as the sum of two component functions, one of which is even and the other odd. (10%)

8. An important property of the Laplace transform is the convolution theorem. State this theorem and prove it. (15%)

1. Solve the following differential equation

$$ydx + (x - \ln y)dy = 0 \qquad (15)$$

2. The following differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$$

exists on the interval $-1 \le x \le 1$ and λ is a real eigenvalue. Is it always true for $y_i(x) \ne y_j(x)$ that $\int_{-1}^{+1} y_i(x)y_j(x)dx = 0$? Why ? (20)

3. Find the position vector $\vec{r}$ of a plane tangent to a surface $x^2 + y^2 + 4z^2 = 4$ at the point $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. (15)

4. Prove that the work done by a gravitational force $\vec{F}(\vec{r})$ is independent of the path C. That is, the integral

$$W = \int_C \vec{F}(\vec{r}) \bullet d\vec{r} \qquad (15)$$

depends only on end points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Where $\vec{F}(\vec{r}) = -\nabla\phi(\vec{r})$, $\phi(\vec{r})$ is a scalar function, $\vec{r}$ is the position vector, and ∇ is the nabla.

5. Let $u_1(t)$ be a solution of the following equation

$\frac{d^2u_1(t)}{dt^2} + \frac{du_1(t)}{dt} + u_1(t) = \delta(t)$ with initial conditions $\frac{du_1(0)}{dt} = 0$ and $u_1(0) = 0$,

where a and b are constants and $\delta(t)$ is the Dirac delta function.

Assume $u_2(t)$ be a solution of the following equation

$\frac{d^2u_2(t)}{dt^2} + \frac{du_2(t)}{dt} + u_2(t) = f(t)$ with the same conditions $\frac{du_2(0)}{dt} = 0$ and $u_2(0) = 0$,

Prove that

$$u_2(t) = \int_0^t f(\tau)u_1(t-\tau)d\tau \qquad (15)$$

(背面仍有題目,請繼續作答)

6. Solve the following wave equaton

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \quad (0 < x, 0 < t) \qquad (20)$$

with the following conditions

$y(x,0) = 0$, $\quad (0 < x)$

$\dfrac{\partial y(x,0)}{\partial t} = e^{-2x}$, $\quad (0 < x)$

$y(0,t) = \sin(3t)$, $\quad (0 < t)$

1. Find the amplitude of resonance for the vibration of a particle governed by

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=P\cos\Omega t$$

where m, c, k, P and Ω are all constants. (20)

2. Solve the following differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=f(x),\quad y(0)=\frac{dy(0)}{dx}=0,\text{ where}$$

$f(x)=5x$, if $0<x<2$ and $f(x)=10$, if $2<x$, (20)

3. Prove that

(a) The eigenvalues of a Hermitian matrix are always real. (10)

(b) The eigenvalues of similar matrices are the same. (10)

4. Calculate the following surface integral

$$I_s=\int_S \vec{F}\bullet\vec{n}dS,$$

where the vector field $\vec{F}=2z\vec{i}+(x-y-z)\vec{k}$,

$\vec{n}$ denotes the unit outer normal vector of the surface S: $z=x^2+y^2$; $x^2+y^2\le 6$, (15)

5. For a wave equation

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial x^2},\quad 0\le t,\ 0\le x$$

(a) Show the D'Alembert's solution of the above equation (10)

(b) Solve $\phi(x,t)$ if $\phi(x,0)=\dfrac{d\phi(x,0)}{dt}=0$ and $\phi(0,t)=[u(t)-u(t-2)](-t^2+2t)$

where $u(.)$ denotes the unit step function. (15)

1. Solve the following differential equation

$$x^3\frac{d^3y}{dx^3}-3x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}-6y=x^4\ln x \qquad (20)$$

2. Given the differential equations as follows

$$(1-x^2)\frac{d^2y_1}{dx^2}-2x\frac{dy_1}{dx}+ay_1=0 \quad \text{and}$$

$$(1-x^2)\frac{d^2y_2}{dx^2}-2x\frac{dy_2}{dx}+by_2=0 \qquad (20)$$

where a and b are constants and $a \neq b$.

Is $\int_{-1}^{1} y_1(x)y_2(x)dx=0$ always true? Why?

3. Define the multiplication of matrices as follows

$$Q=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}3 & -2 & 0\\ -2 & 5 & -3\\ 0 & -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$, where x_1, x_2 and x_3 are any arbitrary real numbers.

Is $Q>0$ always true ? why ? (20)

4. Verify the Green′s theorem for the given vector $\vec{F}=xy\vec{i}+2x\vec{j}$ along a square contour C with vertices at (0,0), (1,0), (1,1) and (0,1) (20)

5. Solve the following wave equation for a string of length L

$$\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}+\delta(x-a)e^{-i\omega t}$$, with the boundary conditions $y(0,t)=y(L,t)=0$,

where $\delta(.)$ is the Dirac′s delta function, $0<a<L$ and ω is a constant. (20)

本試題是否可以使用計算機： ☑可使用 ， □不可使用 （請命題老師勾選）

1. The Bessel function of the first kind is as follows

$$J_\nu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(x/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}$$, where $\Gamma(.)$ is the gamma function.

Prove

(a) $\dfrac{d(x^{-\nu}J_\nu(x))}{dx}=-x^{-\nu}J_\nu(x)$, (10)

(b) $J_{-n}(x)=(-1)^n J_n(x)$, if n is an integer (10)

2. Solve the following differential equation (20)

$\dfrac{d^2y}{dx^2}+y=\delta(x-a)$ with conditions as follows

$y(0)=y(L)=0$, where a is a constant and $0<a<L$,

and $\delta(.)$ is the Dirac's delta function.

3. Given a matrix as follows (20)

$$A=\begin{bmatrix} 6 & -4 & 0\\ -4 & 7 & -3\\ 0 & -3 & 3\end{bmatrix},$$ of which the eigen vectors are

$\begin{Bmatrix}1.0\\0.4205\\x_1\end{Bmatrix}$, $\begin{Bmatrix}x_2\\1.0\\1.1985\end{Bmatrix}$ and $\begin{Bmatrix}2.1047\\x_3\\1.0\end{Bmatrix}$, please find x_1, x_2 and x_3.

4. Calculate the following integral of a vector $\bar{F}=2xy\bar{i}+zy\bar{j}-e^z\bar{k}$ (20)

$$I=\int_C \bar{F}\bullet d\bar{l}$$

where $|d\bar{l}|$ is a line segment of line C described as follows

C: a parabola $y=x^2$, $z=0$, from $(0,0,0)$ to $(2,4,0)$ in the xy-plane.

5. Solve the following partial differential equation (20)

$\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=\sin y$, with the following conditions

$$\phi(0,y)=\phi(1,y)=0,\ \ \phi(x,0)=\frac{\partial\phi(x,\pi/2)}{\partial y}=0$$

本試題是否可以使用計算機：☑可使用　，　□不可使用　（請命題老師勾選）

1. Solve the differential equation $\frac{dy}{dx} = (x+y+1)^2$.[Hint: let $u = x+y+1$] (20)

2. (a) Explain Cauchy-Riemann equations.
   (b) Give the real part $u(x,y) = x^2 - y^2$ of an analytic complex function $f(z) = u(x,y) + iv(x,y)$, find the imaginary part $v(x,y)$.
   (c) Determine the derivative of $f(z)$. (20)

3. (a) Explain half-range Fourier series expansion.
   (b) Expand the function $f(x) = x+1,\ \ 0 < x < \pi$ in a Fourier series and in a Fourier sine series (half-range expansion). (20)

4. (a) Explain the directional derivative of a function.
   (b) Find the directional derivative of the function $f(x,y) = x^2 + y^2$ at point $(3,4)$ in the direction $2\mathbf{i} + \mathbf{j}$.
   (c) Find the maximum directional derivative of the function $f(x,y) = x^2 + y^2$ at point $(3,4)$. (20)

5. Calculate the double integration $\iint_R xy\,dx\,dy = ?$, where $R: \begin{cases} 0 < x+y < 2 \\ 0 < x-y < 2 \end{cases}$.[Hint: let $u = x+y$, $v = x-y$] (20)

1. Consider the second-order homogeneous linear differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0.$$

a) Find the two linearly independent solutions f_1 and f_2 of this equation which are such that

$f_1(0)=1$ and $f_1'(0)=0$

and

$f_2(0)=0$ and $f_2'(0)=1$ (5%)

b) Express the solution

$3e^x+2e^{2x}$

as a linear combination of the two linearly independent solutions f_1 and f_2 defined in (a). (5%)

2. Consider the differential equation

$$(4x+3y^2)dx+2xydy=0$$

a) Show that this equation is not exact.(5%)

b) Find an integrating factor of the form x^n, where n is a positive integer.(5%)

c) Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation. (5%)

3. The function f has at (1,-1) a directional derivative equal to $\sqrt{2}$ in the direction toward (3,1), and $\sqrt{10}$ in the direction toward (0,2).

（背面仍有題目,請繼續作答）

a) Find the value of $\partial f / \partial x$ and $\partial f / \partial y$ at (1,-1).(5%)

b) Determine the derivative of f at (1,-1) in the direction toward (2,3). (5%)

4. Find a unit tangent vector to the curve of intersection of the plane $y - z + 2 = 0$ and the cylinder $x^2 + y^2 = 4$ at the point (0,2,4) (10%)

5. Evaluate the line integral

$$\int_c \frac{-y dx + (x-1) dy}{(x-1)^2 + y^2}$$

where c is any piecewise smooth simple closed curve containing the point (1,0) in its interior. (15%)

6. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(1+x^2)} dx$$

by complex variable methods. (15%)

7. Show that any function f(t) can be expressed as the sum of two component functions, one of which is even and the other odd. (10%)

8. An important property of the Laplace transform is the convolution theorem. State this theorem and prove it. (15%)

成功大學

航空太空工程學系

91~97 學年度
工程數學考古題

1. (18 points)
Consider the differential equation

$$y''(t) + y(t) = r(t)$$

where

$$r(t) = \begin{cases} 0, & \text{if } -2 \le t < 0 \\ 2, & \text{if } 0 \le t < 2 \end{cases}$$

and

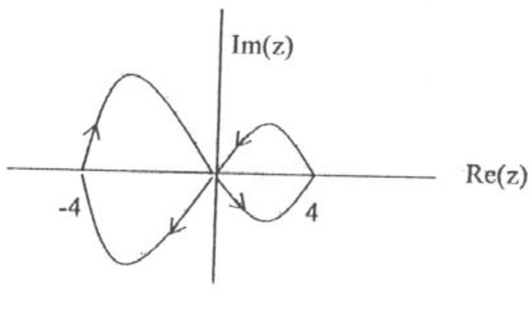
$$r(t+4) = r(t).$$

a). Find the Fourier series of $r(t)$.
b). Find the general solution of the differential equation.

2. (16 points)
Let

$$f(z) = \frac{3z}{(z+2)(z-1)^2}.$$

Evaluate $\int_C f(z)\,dz$ over the following contours:
a). $C = C_1 : |z+1| = 0.5$.
b). $C = C_2$: the boundary of the triangle with vertices 2, $2i$, and $-2i$.
c). $C = C_3$: see the figure.



z-plane

3. (16 points)
a). Find a unit vector perpendicular to the plane $4x + 2y + 4z = -7$.
b). Also, what is the shortest distance between the origin and this plane.

(背面仍有題目,請繼續作答)

4. (18 points)

Solve the following ordinary differential equations:

a). $\frac{dy}{dx} + y = xe^{-x}, \quad y(0) = 1.$

b). $3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 1 = xe^{-x}, \quad y(0) = 1, y(1) = 0.$

c). $\frac{d^2y}{dx^2} + 2y = x, \quad y(0) = 0, y(\sqrt{2}) = 1.$

5. (16 points)

Find the eigenvalues and eigenvectors of the matrix

$$[A] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

6. (16 points)

Use the Laplace transform to solve the following simultaneous equations

$$\frac{d^2y_1}{dt^2} = y_1 + y_2 + \sin(2t)$$

$$\frac{d^2y_2}{dt^2} = -4y_2 + u(t-2)e^{-2t}$$

$$y_1(0) = 1, y_2(0) = 0, \dot{y}_1(0) = \dot{y}_2(0) = 0,$$

where the unit step function $u(t-2)$ is defined as

$$u(t-2) = \begin{cases} 0, & \text{if } t \le 2, \\ 1, & \text{if } t > 2. \end{cases}$$

1. (20%)

Solve the following differential equations for $y(x)$.

a) $y' - y = xy^2$, $y(0) = 0.5$

b) $y'' - y = 4xe^x$, $y(0) = 1$, $y'(0) = 0$

2. (20%)

a) Evaluate the integral

$$\int_C \frac{ie^z}{(z-2+i)^2} dz$$

where C is the counter clockwise circle $|z| = 2$.

b) Consider the mapping, $w = 1/z$. Describe and sketch

i) the image of circles which do not pass through the origin in the z plane.

ii) the image of circles which pass through the origin of the z plane.

3. (20%)

Find the radius of convergence and interval of convergence of the power series:

a) $\sum_{n=0}^{\infty} \frac{1}{n+1}(x+1)^n$,

b) $\sum_{n=0}^{\infty} (-\frac{1}{3})^n (x-1)^n$,

c) $\sum_{n=0}^{\infty} \frac{1}{n3^n}(x+1)^n$.

（背面仍有題目,請繼續作答）

4. (20%)Consider a vertical system of masses and springs. The notations in the figure are:

k_{01}, k_{12}, k_{23}, and k_{34} : spring constants

m_i, i=1,2, and 3: masses

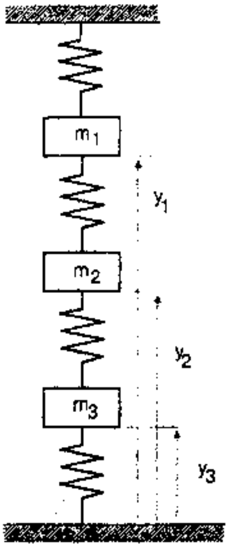
y_i: displacement of mass i from the static position

Assuming there are no frictions, the differential equations for displacements of the masses are given by

$$m_1 \frac{d^2}{dt^2} y_1(t) = -(k_{01} + k_{12}) y_1 + k_{12} y_2$$

$$m_2 \frac{d^2}{dt^2} y_2(t) = k_{12} y_1 - (k_{12} + k_{23}) y_2 + k_{23} y_3$$

$$m_3 \frac{d^2}{dt^2} y_3(t) = k_{23} y_2 - (k_{23} + k_{34}) y_3$$



Derive the eigenvalue problem associated with a harmonic oscillation. Assume that $m_1=m_2=m_3=m$ =constant.

5. (20%)

Use the method of the Fourier sine function expansion to solve the problem:

PDE $\quad u_t = u_{xx} + \sin(3\pi x) \qquad 0 < x < 1$

BCs $\quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$

IC $\quad u(x,0) = \sin(\pi x) \qquad 0 \le x \le 1$.

1. (20%)

(a). Solve the equation $(x+3y)dx-(x-y)dy=0$ with $y(0)=1$,

$\ln\left(\sqrt{x+y}\right)^{-1}=?$

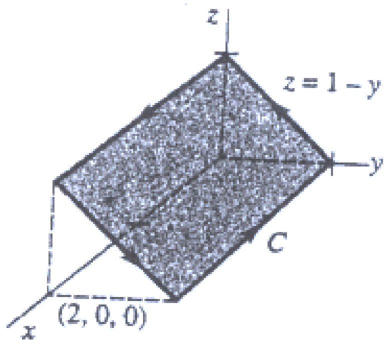
(b). Solve the equation $y'''(x)-y''(x)-y'(x)+y(x)=e^{x}+e^{-x}$ for $y(x)$.

2. (20%)

Let $\vec{F}=z\vec{i}+x\vec{j}+y\vec{k}$. Assume C is the boundery of the plane $z=1-y$ shown in the figure.

(a). $\nabla\times\vec{F}=?$

(b). Evaluate $\oint_C \vec{F}\bullet d\vec{R}$, where $d\vec{R}=dx\vec{i}+dy\vec{j}+dz\vec{k}$ is the differential displacement along C.



(背面仍有題目,請繼續作答)

3. (20%)

Consider the complex functions

a). $f(z) = \dfrac{1}{z^4 - z^5}$,

b). $f(z) = \dfrac{z+1}{z^4 - 2z^3}$..

Integrate $f(z)$ **clockwise** around the circle $C : |z| = 1/2$.

4. (20%)

Use the method of Fourier sine series to solve the problem:

$$u_{tt} = 4u_{xx} \qquad 0 < x < 3, t > 0$$

$$u(0,t) = u(3,t) = 0, \qquad t > 0$$
$$u(x,0) = 0, \qquad 0 < x < 3$$
$$u_t(x,0) = x(3-x), \qquad 0 < x < 3$$

5. (20%)

Let $x = s + t^2$ and $y = s^3 + t$

a) Compute $\dfrac{\partial x}{\partial s}$ and $\dfrac{\partial y}{\partial t}$,

b) Compute $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial t}{\partial x}$,

c) Let $u = xy$, compute $\dfrac{\partial u}{\partial s}$,

d) Let $v = s + t$, compute $\dfrac{\partial v}{\partial x}$.

1. (a) Show that the differential form

$$2\sin 2x \sinh y\, dx - \cos 2x \cosh y\, dy$$

is exact. (5%)

(b) Solve the differential equation (5%)

$$2\sin 2x \sinh y\, dx - \cos 2x \cosh y\, dy = 0.$$

(c) Solve the initial value problem. (10%)

$$y'' - y = 2\cos x,\ y(0) = 0,\ y'(0) = 3$$

2. (a) Calculate the integral (10%)

$\int_C \vec{F}(\vec{r}) \cdot d\vec{r},\ \vec{F} = [2z,\ 1,\ -y],\ C : \vec{r} = [\cos t,\ \sin t,\ 2t]$ *from* $(0,\ 0,\ 0)$ *to* $(1,\ 0,\ 4\pi)$.

(b) Evaluate the integral (10%)

$$\int_{(\pi/2,\ -\pi)}^{(\pi/4,\ 0)} (\cos x \cos 2y dx - 2\sin x \sin 2y dy).$$

(20pt) 3. Solve $y''' - y' = \sin t$ by Laplace transform, given $y(0) = 2$, $y'(0) = 0$, and $y''(0) = 1$. Note that $L\left\{\frac{\sin \omega t}{\omega}\right\} = \frac{1}{s^2 + \omega^2}$ and $L\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$.

(背面仍有題目,請繼續作答)

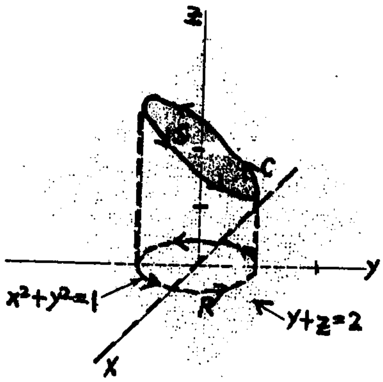
4. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.
(20pt)

(a). The determinant $\det(-A^{10}) = ?$
(b). The eigenvalues $\lambda(A^{10}) = ?$
(c). The rank $rank(A) = ?$
(d). How many linearly independent eigenvectors does matrix A have ?

5. Let $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$. Assume that C is the trace (邊緣) of the cylinder $x^2 + y^2 = 1$ in the plane $y + z = 2$. Orient C counterclockwise as viewed as from above. See the following figure.
(20 pt)

(a). Calculate the surface integral $I_1 = \iint_S (\nabla \times \vec{F}) \bullet \vec{n} dS$ directly, where $\vec{n}$ is the outward unit normal of the surface S enclosed by C, and dS is the differential area element of the surface S. (Note that you are prohibited to evaluate I_1 by using the result in (b).)

(b). Calculate the line integral $I_2 = \oint_C \vec{F} \bullet d\vec{R}$ directly.
where $d\vec{R} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ is the differential displacement along C.
(Note that you are prohibited to evaluate I_2 by using the result in (a).)



1. a). Given three vectors

$$\vec{a} = [a_1, a_2, a_3], \vec{b} = [b_1, b_2, b_3], \vec{c} = [c_1, c_2, c_3],$$

compute $\vec{a} \cdot (\vec{b} \times \vec{c})$ and **explain** the geometrical interpretation of the scalar triple product. (7%)

b). Given the curve: $\vec{r}(t) = t\,\vec{i} + \cosh t\,\vec{j}$, $t \in [0,1]$, **compute** the length of $\vec{r}(t)$ from $t = 0$ to $t = 1$, i.e., $\int_0^1 \sqrt{\vec{r}'(t) \cdot \vec{r}'(t)}\,dt$, and **explain** the geometrical meaning of $\vec{r}'(t)$. (7%)

c). Find the directional derivative of $f(x,y,z) = xyz$ at $P(-1,1,3)$ in the direction of $\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}$. (6%)

2. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

a). What is Cayley-Hamilton theorem ? Give a simple example to demonstrate it. (5%)

b). $A^{30} = ?$ (5%)

c). What is the determinant of A^{30} ? (5%)

d). How many linearly independent eigenvectors does matrix A have ? (5%)

3. Evaluate the following integrals:

$$I_1 = \int_0^\infty \frac{\cos ax}{x^2+1}\,dx \quad \text{and} \quad I_2 = \int_0^\infty \frac{\sin ax}{x^2+1}\,dx \quad (a > 0)$$

by using the *Residue Theorem* in the complex variables theory. (20%)

4. Use the Fourier series method to solve the problem: (20%)

$$u_t = 3u_{xx} \qquad 0 < x < 2, t > 0$$

with

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

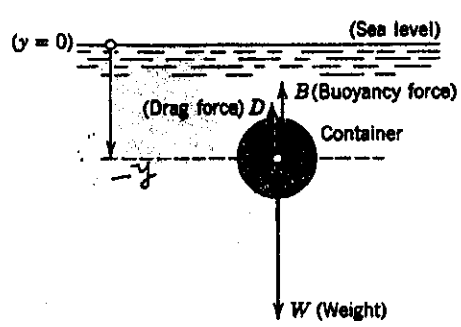
$$u(x,0) = 2[1 - \cos(\pi x)], \qquad 0 < x < 2$$

（背面仍有題目,請繼續作答）

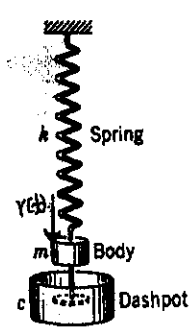
5. The Newton second law of motion: the equivalent force of mass times acceleration is equal to resulting force of weight, buoyant force, and drag force (say, $ma = \sum \text{force}$).

a). A spherical ball of weight 1 [kg] is immerged in water and moving upward. If the buoyancy force is 1 [nt], and the drag force is $D = \alpha \cdot V$, where $\alpha = -0.1$ [nt·s/m] and V is the velocity. The initial velocity is $V_0 = 10$ [m/s], and initial location is at $y(0) = -200$ [m]. The gravitational acceleration is 9.8 [m/s^2]. Find the velocity distribution with respect to time. (10%)

Note: [nt] = [Newton], 1 [nt] = 1 [kg·m/s^2].



b). Consider the simplified mass-spring system, with mass $m = 1$ [kg], spring constant $k =$ 20 [nt/m], the damping constant is $c =$ 4 [kg/s], input force is $r(t) = 0.1 \times \cos(4t)$ [nt]. If the initial location of mass is at $y = 0$ [m], find the location of the mass with respect to time. (10%)



本試題是否可以使用計算機： ☑可使用 ， ☐不可使用 （請命題老師勾選）

20%[1]

(a). Find the general solution to the following equation: $\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}-2y=3e^{2t}$.

(b). Find the solution to the above equation for $y(0)=3$ and $\left.\dfrac{dy}{dt}\right|_{t=0}=4$.

[2]

Solve the system of equations $\dot{x}(t)=Ax(t)+B(t)$,

where $A=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$, $B(t)=\begin{bmatrix}-2\\ -2\end{bmatrix}$, $x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$, $x(0)=\begin{bmatrix}2\\ 2\end{bmatrix}$

5%(a). What are the eigenvalues of matrix A ?

15%(b). What is the solution $x(t)$ to the system ?

20%[3]

(a). Find the coordinates (x, y, z) of the point on the plane

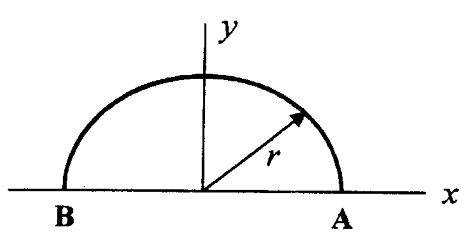
$$ax+by+cz=d \quad \text{with } a^2+b^2+c^2=1,$$

which is closest to the origin. Also find the shortest distance from the origin to the plane.

(b). Evaluate the line integral

$$I=\int_C f\,ds,$$

where $f=x^2y$ and C is the semicircular arc of radius r from A to B as shown in the figure.



（ 背面還有題目，請繼續作答 ）

20%[4]

Use the Fourier series method to solve the problem:

$$u_t = 4u_{xx} \qquad 0 < x < 2, t > 0$$

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 2 - 2\cos(\pi x) + 4\sin(2\pi x), \qquad 0 < x < 2$$

[5]

12% (a). State the sufficient and necessary conditions such that a complex function is analytic. How about the relation between these conditions and the uniqueness of the derivative $f(z)$? Please use $f(z) = \dfrac{1}{1-z}$ to explain your answer. Is this $f(z)$ analytic at the point $z = 1$? Is this $f(z)$ analytic at points satisfy $|z| > 0.001$?

8% (b). Find the complex integration $\oint_C \dfrac{1}{1-z} dz = ?$ along the following closed curves in the counterclockwise sense.

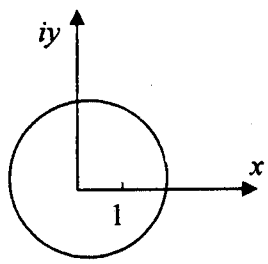


Figure 1

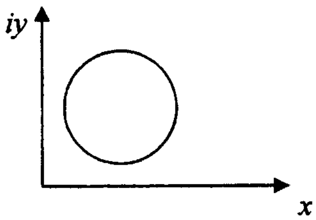


Figure 2

1. Vector Analysis:

(a) Let $F(x,y,z)=(y,2xz,ze^x)$, compute div F and curl F. (7%)

(b) Let $f(x,y,z)=x^2y\cos(yz)$, compute grad f. (6%)

(c) Evaluate the line integral (7%)

$$\int_C x dx - xy dy$$

If C is given by $x=t^2, y=-t;\quad 1\le t\le 2.$

2. Find the general solution of the following differential equations:

(a) $y^2dx+(2xy-x^4)dy=0$ (6%)

(b) $y''-2y'+y=e^x+x$ (7%)

(c) $\mathbf{x}'=\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}\mathbf{x}+\begin{bmatrix}-8\\ 3\end{bmatrix}$ (7%)

3. Given a $n\times n$ matrix A we wish to find a $n\times 1$ vector X so that $AX=Z$ for a given $n\times 1$ vector Z. List all conditions on A and Z in order that a solution may exist. (20%)

4. (20 points)

a). Let

$$f(z)=\frac{z^{1/2}}{(z+1)(z-2)^2},$$

please locate all the singularities and determine their nature.

b). Evaluate the integral

$$\frac{1}{2\pi i}\int_{|z|=3}\frac{dz}{(z^2+1)(z-2)^2(z-4)}.$$

（背面仍有題目,請繼續作答）

5. (20%) With the help of the schematic diagram, solve the following two-dimensional boundary value problem within the square in term of the method of separation variable.

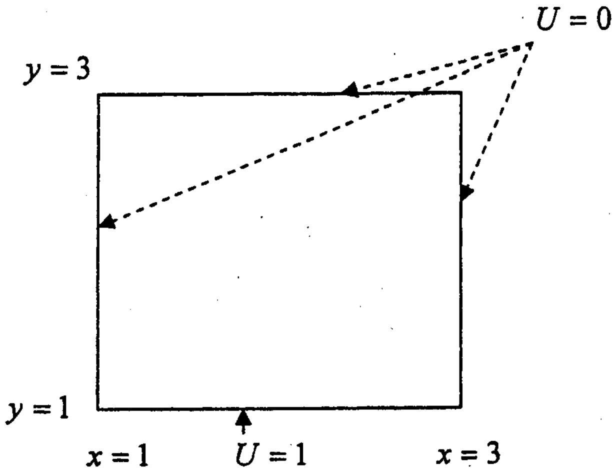
$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0$$

$U=1,\ 1<x<3, y=1$

$U=0,\ 1<x<3, y=3$

$U=0,\ x=1, 1<y<3$

$U=0,\ x=3, 1<y<3$



成功大學

資源工程研究所

91~97 學年度
工程數學考古題

1. (10%)
若 $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+4y=0$，求 $y(x)=?$

2. (10%)
若 $\dfrac{dx}{dt}+15x+10y-60=0$，$\dfrac{dx}{dt}-2\dfrac{dy}{dt}+5x-10y=0$，　$x(0)=0$,　$y(0)=0$,
求 $x(t)=?$　$y(t)=?$　$t>0$

3. (10%)
一質量爲 m 之物體在傾角爲θ之斜面上，由靜止開始下滑：(列出方程式即可，不需求解)
(a) 若不計空氣阻力，但考慮物體與斜面之摩擦力(摩擦係數爲μ)，試列出控制該物體速度(v)之方程式?
(b) 若空氣阻力之大小與速度成正比(阻力係數爲κ)且一併考慮物體與斜面之摩擦力，試列出控制該物體速度(v)之方程式?

4. (10%)
若 $f(x,y)=x\sin(y^2)+y\cos(xy)$，　求 $\dfrac{\partial^2 f}{\partial x\partial y}=?$　$\dfrac{\partial^3 f}{\partial x\partial y^2}=?$

5. (10%)
若 $z(x,y)=\sqrt{x^2+y^2}$，$x(t)=e^t$，$y(t)=\sin t$，求 $\dfrac{dz}{dt}=?$

6. (10%)
若一心臟線之極座標表示爲 $r=2(1+\cos\theta)$，求此心臟線所包絡之面積?

7. (10%)
求 (10, 0)至 $y=4x^2$ 之最短距離?

8. (10%)
求 $\vec{F}=(y+z)\vec{i}+(z+x)\vec{j}+(x+y)\vec{k}$ 沿著曲線 C: $t\vec{i}+t^2\vec{j}+t^3\vec{k}$, $0<t<1$ 的線積分 $\int_C \vec{f}\cdot d\vec{R}=?$

9. (10%)
求 $\begin{bmatrix}1 & -2 & 1\\ -1 & 3 & 2\\ 2 & -2 & 7\end{bmatrix}$ 之逆矩陣?

10. (10%)
若 $2\dfrac{\partial u}{\partial x}+u=0$，求 $u(x,y)$ 之通解?

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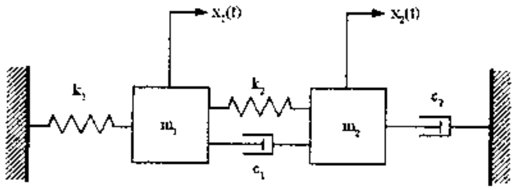
注意：1. 必須列出計算過程，只列答案不予計分。 2. 禁止使用程式型計算機。

1. 15%	若 $m\dfrac{d^2y(t)}{dt^2}+c\dfrac{dy(t)}{dt}+ky(t)=0$, 初始條件：$y(0)=0$, $\dfrac{dy(0)}{dt}=0$ 其中 m, c, k 爲已知正實數，解 $y(t)$ 並依據 m, c, k 間之關係繪出 $y(t)$ 對 t 之示意圖？
2. 20%	導出下圖之運動方程式並寫出其 Laplace 變換？(列式即可，不需求解。)  初始條件： $x_1(0)=\alpha$, $\dfrac{dx_1(0)}{dt}=\beta$, $x_2(0)=\gamma$, $\dfrac{dx_2(0)}{dt}=\delta$
3. 20%	(a)求基本週期爲 2π之函數：$f(x)=\begin{cases}-1, & -\pi<x\le 0\\ 1, & 0<x\le\pi\end{cases}$ 之傅立葉級數(Fourier series)展開？(b)以上述問題爲例說明在 $x=\pi$處，傅立葉級數會收斂至何值？(c)扼要說明 Gibbs 現象？
4. 15%	若 $\begin{bmatrix}4&0&2&1\\2&4&-1&-2\\3&2&0&5\\1&3&-2&0\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=\begin{Bmatrix}7\\10\\2\\3\end{Bmatrix}$ 利用 Gauss 消去法將之化簡爲： (a) $\begin{bmatrix}1&u_{12}&u_{13}&u_{14}\\0&1&u_{23}&u_{24}\\0&0&1&u_{34}\\0&0&0&1\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=\begin{Bmatrix}c_1\\c_2\\c_3\\c_4\end{Bmatrix}$ (b) $\begin{bmatrix}4&0&2&1\\2&4&-1&-2\\3&2&0&5\\1&3&-2&0\end{bmatrix}=\begin{bmatrix}\ell_{11}&0&0&0\\\ell_{21}&\ell_{22}&0&0\\\ell_{31}&\ell_{32}&\ell_{33}&0\\\ell_{41}&\ell_{42}&\ell_{43}&\ell_{44}\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}&u_{14}\\0&1&u_{23}&u_{24}\\0&0&1&u_{34}\\0&0&0&1\end{bmatrix}$ ，求 u_{ij}, x_i, c_i, ℓ_{ij} ?
5. 10%	求 $[A]=\begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$ 之特徵值及特徵單位向量？
6. 10%	計算 $\int_0^1 x^{0.7}\ln x\,dx=$?(令 $J(a)=\int_0^1 x^a dx=\dfrac{1}{a+1}$, $a>-1$，並利用 Leibniz 公式。)
7. 10%	(a)求 $yz-\dfrac{x}{yz^2}$ 在 $(2,1,-1)$ 處與 x, y, z 座標正向夾相等角度方向之方向導數 728 (directional derivatives)？(b)求 $yz\bar{i}+xz\bar{j}-xy\bar{k}$ 在 $(2,1,-1)$ 處之旋度(curl)？

93學年度國立成功大學研究所招生考試 資源工程學系 甲組 乙 工程數學 試題 共 1 頁 第 1 頁

注意：1. 第 1 題請在答案卷上繪製相同的空格作答。2. 禁止使用程式型計算機。

1. **15%**

方程式	自變數	因變數	幾階	幾元	常？	線性？	幾次
$y^{(iv)}+4x^2y=y^3$	x	y					
$u_x+v_y=x^2$, $u_y-v_x=\cos y$	x, y	u, v					
$v_x+x\,v_y=v_{tt}$	x, y, t	v					

2. **15%**

(a) 若 $t^2\dfrac{d^2y(t)}{dt^2}-t\dfrac{dy(t)}{dt}+y(t)=0$，求 $y(t)$？(b) 若 $\dfrac{d^2y(x)}{dx^2}+y=\sin x$，求 $y(x)$？

(c) 若 $\dfrac{dx(y)}{dy}=x+y+2$，求 $x(y)$？

3. **20%**

(a) Laplace 變換的條件為何？(b)求 $f(t)=\begin{cases}1, & 1\le t\le 2\\ 0, & elsewhere\end{cases}$ 之 Laplace 變換？

(c) Fourier 變換的條件為何？(d))求 $f(t)=\begin{cases}e^{-t}, & t\ge 0\\ 0, & t\le 0\end{cases}$ 之 Fourier 變換？

4. **10%**

若 $\dfrac{\partial u(x,y)}{\partial x\partial y}=0$，$0\le x,y\le 1$，且 $u(x,0)=x(1-x)$，$u(0,y)=y$，$u(1,y)=y$，求 $u(x,y)=$？

5. **15%**

若 $[A]=\begin{bmatrix}3 & 2 & 0 & 0\\ 6 & 8 & 0 & 0\\ 0 & 0 & 4 & 7\\ 0 & 0 & 2 & 5\end{bmatrix}$，(a)求[A]之秩(rank)？(b)求[A]之跡(trace)？(c)求[A]之反矩陣？

6. **10%**

(a)求以(-1, 0, 1), (2,-1,4), (2,1, 5), (-2, 1, 4)為四頂點之四面體體積？

(b)求從(-2, 0, 1)到(-1, 0, 1), (0,-1,1), (1,1, -2)三點所成平面之最短距離？

7. **15%**

求 $\dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$ 在(1, 1, 1)處之(a)散度(divergence)？及(b)旋度(curl)？

(c)求 $\sin(x+2y+3z)$ 在$(2\pi, -\pi, 0)$處往什麼方向其位置變化率為最大？

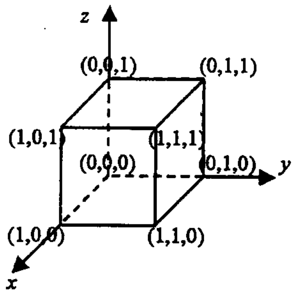
資源工程數學(甲)：1.禁用計算機。2.須列計算過程只列答案不計分。3.第 9 題請在答案卷上繪製相同空格作答。

1. 10%　解 $\dfrac{dy}{dx}=\dfrac{y}{x}+y^2$？

2. 10%　解 $\dfrac{d^2y}{dt^2}+y=\cos t-\sin t$？

3. 10%　解 $x(t)$　$t\ge 0$：$x''(t)-x(t)=\begin{cases}0, & 0\le t<1\\ 1, & 1\le t<\infty\end{cases}$　$x(0)=2$　$x'(0)=0$

4. 10%　$\vec{F}=y\vec{i}+x\vec{j}+z\vec{j}$，S 爲如圖方塊區域之包絡表面，計算 $\iint_S \vec{F}\cdot\vec{n}dA$？其中 $\vec{n}$ 爲單位外法向量，dA 爲表面積單元。



5. 10%　求矩陣 $\begin{bmatrix}1 & 2 & 2\\ -1 & -2 & -1\\ 1 & 1 & 0\end{bmatrix}$ 之特徵值及特徵向量？舉例說明特徵值問題在工程及物理的應用？

6. 10%　若 $u(x,y,z)=\sin(xyz)$，計算(a) u 在 $(\pi,1,1)$ 處往正 x 軸方向之方向導數？(b) u 在 $(1,\pi,1)$ 處方向導數之最大值及其方向？

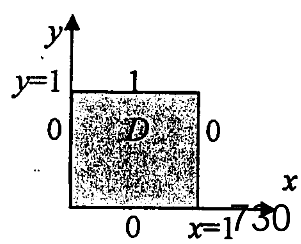
7. 10%　對 $f(x)=x,\ 0\le x\le 1$，進行 Fourier 半幅餘弦級數(half range cosine)展開？

8. 10%　對 $f(x)=\begin{cases}1, & -1\le x\le 1\\ 0, & \text{elsewhere}\end{cases}$，進行 Fourier 變換 $F(\omega)$？並求 $|F(\omega)|$？

9. 10%

方程式 (equation)	自變數 (independent variable)	因變數 (dependent variable)	階？(order)	類型？(type)	齊次？(homogeneous)	線性？(linear)
$u_t+u_x=u_{xt}$	x, t					
$\nabla^2 p=e^{1/(xyz)}$	x, y, z					

10. 10%　解 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ in D，已知邊界條件：$u(0,y)=0,\ 0<y<1;\ u(1,y)=0,\ 0<y<1;$ $u(x,0)=0,\ 0<x<1;\ u(x,1)=1,\ 0<x<1$。



資源工程數學(乙)：1.禁用計算機。2.須列計算過程只列答案不計分。3.第 9 題請在答案卷上繪製相同空格作答。

1. 10%　解 $\dfrac{dy}{dx}=xy+y^3$？

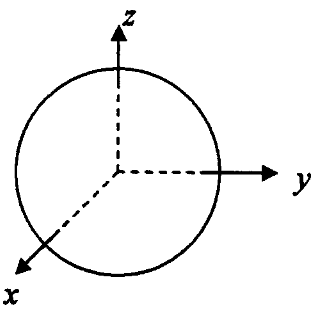
2. 10%　解 $\dfrac{d^2y}{dt^2}-y=e^t+e^{-t}$？

3. 10%　解 $x(t)$　$t\ge 0$：$x''(t)+x(t)=\begin{cases}0, & 0\le t<\pi/4\\ 1, & \pi/4\le t<\infty\end{cases}$　$x(0)=1$　$x'(0)=0$

4. 10%　求矩陣 $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ 之特徵值及特徵向量？並舉例說明特徵值問題在工程及物理的應用？

5. 10%　若 $u(x,y,z)=e^{-xyz}$，計算(a) u 在(1,1,1)處往正 x 軸方向之方向導數？(b) u 在(0,1,0)處方向導數之最大值及其方向？

6. 10%　$\vec{F}=x\vec{i}+y\vec{j}+z\vec{j}$，S 爲如圖球區域之包絡面 $x^2+y^2+z^2=1$，計算 $\iint_S \vec{F}\cdot\vec{n}dA$？其中 $\vec{n}$ 爲單位外法向量，dA 爲表面積單元。



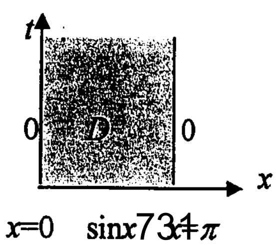
7. 10%　對 $f(x)=1,\ 0\le x\le 1$，進行 Fourier 半幅正弦級數(half range sine)展開？

8. 10%　對 $f(x)=\begin{cases}1, & -1\le x\le 1\\ 0, & \text{elsewhere}\end{cases}$，進行 Fourier 積分？

9. 10%

方程式 (equation)	自變數 (independent variable)	因變數 (dependent variable)	階？(order)	類型？(type)	齊次？(homogeneous)	線性？(linear)
$u_t+u_x=u_{xt}$	x, t					
$\nabla^2 p=e^{1/(xyz)}$	x, y, z					

10. 10%　解 $u(x,t), 0<x<\pi, t\ge 0$，已知：$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t},\ 0<x<\pi,\ t>0$，
$u(0,t)=u(\pi,t)=0, t\ge 0$，$u(x,0)=\sin x,\ 0\le x\le \pi$。



本試題是否可以使用計算機：☐可使用 ， ☑不可使用 （請命題老師勾選）

1. (a)解 $x\dfrac{dy(x)}{dx}-y(x)=2x$ ？(5%)

   (b)解 $\dfrac{d^2y(x)}{dx^2}+16y(x)=\cos 4x$ ？(5%)

   (c)解 $\dfrac{\partial^2 u(x,y)}{\partial x^2}=\dfrac{\partial u(x,y)}{\partial y}$ ？(5%)

2. (a)求 e^{at} 之拉普拉斯變換？其中 a 爲常實數。(5%)

   (b)求 $\sin(bt)$ 之拉普拉斯變換？其中 b 爲常實數。(5%)

   (c)求 $\dfrac{1}{(s-1)(s+3)}$ 之反拉普拉斯變換？(5%)

3. 解釋爲何矩陣加法與純量對矩陣的乘法合乎線性空間之定義？(15%)

4. 若 $[A]=\begin{bmatrix}0 & 1 & 1\\ 0 & -1 & 1\\ 1 & 1 & 0\end{bmatrix}$，計算 $[A]^6$ ？(15%)

5. 推導傅立葉級數(Fourier series)的公式： $f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\dfrac{n\pi x}{L}+b_n\sin\dfrac{n\pi x}{L}\right)$

   其中 $a_0=\dfrac{1}{2L}\int_{-L}^{L}f(x)dx$ ， $a_n=\dfrac{1}{L}\int_{-L}^{L}f(x)\cos\dfrac{n\pi x}{L}dx$ ， $b_n=\dfrac{1}{L}\int_{-L}^{L}f(x)\sin\dfrac{n\pi x}{L}dx$ ，

   $f(x)$有何限制條件？(15%)

6. 說明力學之質量(m)-阻尼(c)-彈簧(k)系統與電學之電阻(R)-電容(C)-電感(L)系統二者間之等效性？請繪圖說明並列出二系統之控制方程？(15%)

7. 敘述高斯散度定理？(10%)

本試題是否可以使用計算機： □可使用 ， ☑不可使用 （請命題老師勾選）

1. (a) $\dfrac{dy}{dx}+y=x$，求一般解 $y(x)=$？(6%)

(b) $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=0$，求一般解 $y(x)=$？(6%)

(c) 求解 $y(t)=1+\int_0^t (t-\tau)y(\tau)d\tau$ ？(6%)

(d) 求解 $\dfrac{dy(t)}{dt}+y(t)=u(t-2)$ ， $y(0)=0$ ， $\dfrac{dy}{dt}(0)=0$ ？(6%)

其中 $u(t-2)=\begin{cases}0 & 0<t<2\\ 1 & 2\le t\end{cases}$

(e) $\dfrac{\partial^2 u(x,y)}{\partial x^2}-y^2u(x,y)=y\sin x$，求一般解 $u(x,y)=$？(6%)

2. 若 $u=(x+2y+3z)^{-1}$， $\vec{s}=\vec{i}+\vec{j}+\vec{k}$，計算

(a) $\vec{\nabla}u$ ？(5%)

(b) $\vec{\nabla}\cdot\vec{\nabla}u$ ？(5%)

(c) $\dfrac{du}{ds}$ at $(2,0,1)$ ？(5%)

(d) 在(2, 0, 1)處 u 在那一個方向之方向導數有極值？(5%)

3. (a) 對 $f(x)=x,\ 0\le x\le 1$，進行 Fourier 半幅正弦級數(half range sine)展開？(10%)

(b) 對 $f(x)=\begin{cases}1, & -1\le x\le 1\\ 0, & \text{elsewhere}\end{cases}$，進行 Fourier 變換？(10%)

4. 求 $[A]=\begin{bmatrix}2&0&0\\0&1&1\\0&1&1\end{bmatrix}$ 之

(a)行列式？(5%)
(b)秩(rank)？(5%)
(c)反矩陣？(5%)
(d)特徵值及特徵單位向量？(10%)
(e) $[A]^{10}$ ？(5%)

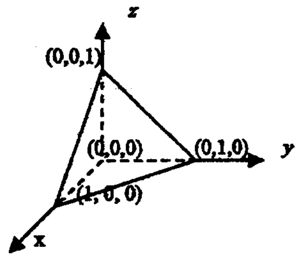
1. 解$\dfrac{d^2y(t)}{dt^2}+y(t)=\cos t$？(7%)

2. 解$x'(t)+x(t)=\begin{cases}0, & 0\le t<1\\ 1, & 1\le t<\infty\end{cases}$　　$x(0)=0$？(7%)

3. 解$y(t)=1+\int_0^t (t-\tau)y(\tau)d\tau$？(7%)

4. $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$，S 爲如圖四面體之包絡表面，計算$\iint_S \vec{F}\cdot\vec{n}dA$？

   其中$\vec{n}$爲單位外法向量，dA爲表面積單元。(10%)



5. 若$u=\sin(x+y+z)$，$\vec{s}=\vec{i}+\vec{j}+\vec{k}$，計算(a)$|\vec{\nabla}u|$在(1, 1, 1)處之極值？(b)$u$在$(2\pi,0,\pi)$處往$\vec{s}$之方向導數？(c)$\vec{\nabla}\times\vec{\nabla}u=$？(15%)

6. 對$f(x)=x,\ 0\le x\le 1$，進行 Fourier 1/4 幅正弦級數與 1/4 幅餘弦級數(quarter range sine & cosine)展開？(12%)

7. 對$f(x)=\begin{cases}1, & -1\le x\le 1\\ 0, & \text{elsewhere}\end{cases}$，分別進行(a)Fourier 變換？(b)Fourier 積分？(12%)

8. (a)求$[A]=\begin{bmatrix}-3&1&-1&1&1\\0&1&1&0&4\\0&0&-3&2&1\end{bmatrix}$之列簡化梯狀矩陣(row reduced eschelon matrix)？(6%)

   (b)求[A]之秩？(4%)

   (c)求$[A]=\begin{bmatrix}-3&1&-1&1&1\\0&1&1&0&4\\0&0&-3&2&1\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$之解空間(solution space)？(6%)

   (d)由(c)求該解空間之維度？(4%)

9. 解$\dfrac{\partial^2u}{\partial x^2}=\dfrac{\partial u}{\partial t}$，已知條件：u(0, t) = 0, 0 < t; u(1, t) = 0, 0 < t; u(x, 0) = x(1-x), 0 < x < 1. (10%)

成功大學

機械研究所

91~97 學年度
工程數學考古題

(甲,乙,丙,丁,戊组)

1. a) 請定義何謂自伴微分演算子(Self-adjoint differential operator)? 並列出正則史敦-利奧比系統(Regular Sturm-Liouville system)微分方程式和其邊界條件?(4%)

b) 正則史敦-利奧比系統微分方程之解,具何種性質?請說明(不必証明),並定義何謂正交函數系統之完整性條件(Complete orthogonal system)? (8%)

c) 求微分方程

$$\frac{d^2y}{dx^2} + \lambda y = 0 ,\quad \text{其邊界條件 } y(0)=0,\ y(l)=0$$

証明其特徵解在 $[0, l]$ 之間為正交函數?

如果為正交函數,請以此正交函數系統將函數 $f(x)=x^2 \quad 0 \le x \le l$. 展開為 Fourier Series.

(13%)

2-1. Evaluate the following integrals: (8%)

(a) $\int_0^\infty (x+1)^2 e^{-x^3} dx$

(b) $\int_0^\infty \frac{x^c}{c^x} dx$

2-2. Let $f(x) = x$ for $0 \le x \le 1$.

(a) Expand f(x) in a Fourier cosine series for period 2. (6%)

(b) Expand f(x) in a Fourier sine series for period 2. (6%)

(c) Explain the relation of the solutions obtained from (a) and (b). (5%)

(背面仍有題目,請繼續作答)

3、求解下列聯立方程組 (25%)

$$\begin{cases} x_1 - x_3 + 2x_4 + x_5 + 6x_6 = -3 \\ x_2 + x_3 + 3x_4 + 2x_5 + 4x_6 = 1 \\ x_1 - 4x_2 + 3x_3 + x_4 + 2x_6 = 0 \end{cases}$$

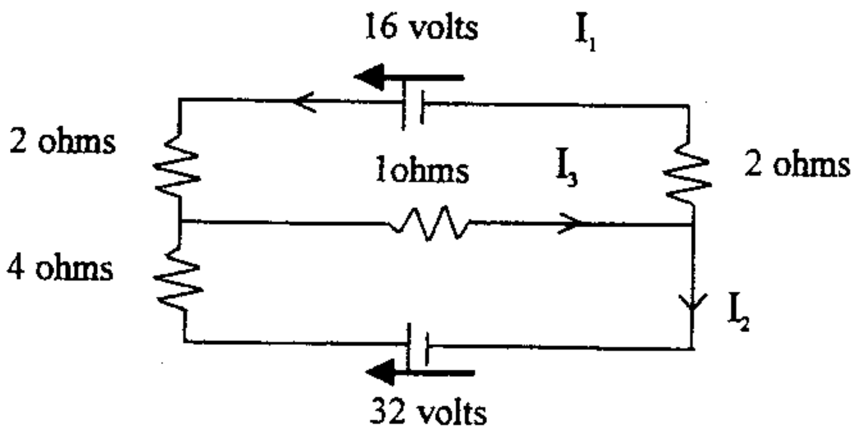
4-1. (10%) Classify the singularities of $f(z) = \dfrac{1}{(z-1)(z-2)}$. Obtain the Laurent expansion centered on $z = 0$ for the three regions: (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

4-2. (15%) Sometimes it is possible to find a physically interesting solution to a partial differential equation by assuming that the solution is a function of a single variable rather than two or more variables. In the particular case of the heat conduction equation

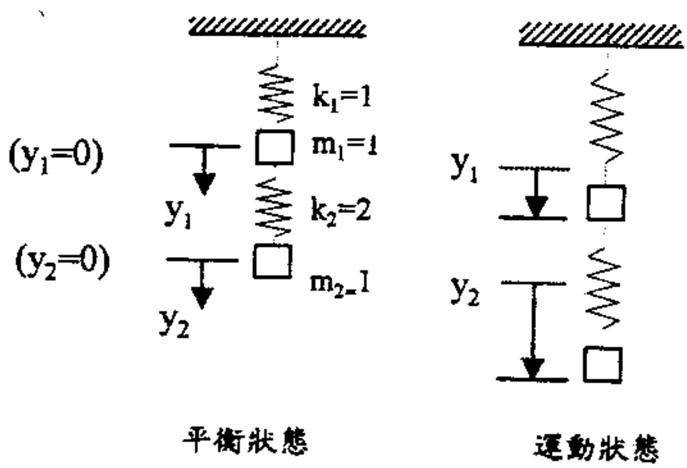
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

try a solution of the form $u(x,t) = f(\xi)$ with $\xi = xt^{-\alpha}$. For what value of α does this work, and what is the result for $f(\xi)$ in that case?

(1).(15%)試以高斯消去(Gauss Elimination)法，求解下列線性電路系統之電流量 I_1,I_2 與 I_3。



(2).(15%)試求解下列線性彈簧－質點系統之質點位移 y_1 與 y_2。



(3).(15%) Find the steady-state oscillation of

$$y'' + 0.02y' + 25y = r(t)$$

where r(t) is given as

$$r(t)=\begin{cases} t+\frac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\frac{\pi}{2} & \text{if } 0<t<\pi \end{cases}$$

here

$r(t+2\pi)=r(t)$

(4).(20%) Let us first consider the temperature (T) in a long thin bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis and is perfectly insulated laterally.

(a) Write down the differential equation for the temperature

(b) Derive the general solution of this differential equation

(c) Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature 0, assuming that the initial temperature is

$$T(0,x) = \begin{cases} x & \text{if} \quad 0 < x < L/2 \\ (L-x) & \text{if} \quad L/2 < x < L \end{cases}$$

(5).(15%)Classify each of the following differential equations by stating the order, whether the equation is homogeneous or non-homogeneous, and it is linear or nonlinear (in which variable.)

(a) $d^2y/dx^2 + 3x^2 = 2(dy/dx)^2$

(b) $dy/dx + y/x = xy^2$

(c) $dy/dx = (x+y)/(x-y)$

(d) $(3x^2 + y\cos x)dx + (\sin x)dy = 0$

(e) $d(yu) = y^2 du$

(6).(10%)Solve $dy/dx = (y + x^4)/x$

(7).(10%)Let S be a closed regular surface and $\vec{r}$ denote the position vector of any point (x, y, z) measured from an origin O. Evaluate

$$\iint_S \frac{\vec{n}\cdot\vec{r}}{r^3}\,dS$$

in which $\vec{n}$ is the outward unit normal vector to dS and $r = |\vec{r}|$.

93學年度國立成功大學研究所招生考試 機械工程學系甲乙丙組丁戊 工程數學 試題 共1頁 第1頁

(1) (10%)State whether the method of Taylor series can be used to solve the differential equation

$$(x^2-3x+2)\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0$$

with

$$y(1)=\frac{dy}{dx}(1)=0$$

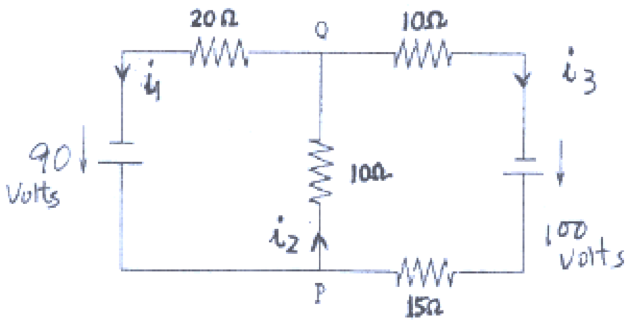
(2) (15%)Evaluate $\oint_c \cot z\,dz$ where c is the circle $|z|=1$ traversed clockwise.

(3) (15%) $\int_0^\infty ye^{-y^2}\sin(2y)dy = ?$

(4) (10%)Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these two functions?

5.(15%) 試估算 $I=\iint_S (x^3dydz+x^2ydzdx+x^2zdxdy)$，其中 S 為圓柱體 $x^2+y^2=a^2$，$0\le z\le b$ 之外包表面。

6.(15%) 試以高斯消去(Gauss Elimination)法，求解下列線性電路系統之電流量。



7.(20%) 一水槽中裝有含 160 公克鹽量之水溶液共 1000 立方米，假設每單位時間有 40 立方米的海水流入槽中，並均勻混合。海水中每立方米含鹽量為(1+cost)公克，而槽中水溶液的流出率為每單位時間 40 立方米，試問槽中水溶液在任意t時間的含鹽量 $y(t)$ 為何？

Problem 1 (5%)

Two different chemical solutions are pumped into a container of volume 100 ℓ, each at the rate of 5 ℓ/sec, and thoroughly mixed solution is pumped out of the container at the rate of 10 ℓ/sec. The inflow concentration of Chemical 1 is q_1 (kg/ℓ) and the inflow concentration of Chemical 2 is q_2 (kg/ℓ). Denote the mass of Chemical 1 (in the container) by x_1 (kg) and the mass of Chemical 2 by x_2 (kg). A catalyst in the container transforms Chemical 1 into Chemical 2 at the rate of $0.4x$ (kg/sec).

(a) Formulate the 2×2 linear system of first order ODEs that x_1 and x_2 satisfy.

(b) Without finding the general solution, determine the steady-state solution for x_1 and x_2.

Problem 2 (10%)

A solution to the unforced equation $\ddot{x} + b\dot{x} + 4x = 0$ is observed to take the value $x = 0$ for $t = 0$ and next at $t = \pi$. (Assume that $b \geq 0$.)

(a) Is the equation underdamped, critically damped, or overdamped? Explain.

(b) What is the value of the damping constant b?

Problem 3 (10%)

A certain matrix $\mathbf{A}$ has eigenvalues 1 and -1, with eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{respectively.}$$

(a) Find the solution for the initial value problem

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} \quad \text{with} \quad \mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) Calculate $\mathbf{A}^{9999}$.

Problem 4 (10%)

The temperature T at a point (x, y, z) in space is inversely proportional to the square of the distance from (x, y, z) to the origin. It is known that $T(0, 0, 1) = 500$.

(a) Find the rate of change of T at $(2, 3, 3)$ in the direction of $(3, 1, 1)$.

(b) In which direction from $(2, 3, 3)$ does the temperature T increase most rapidly?

(c) At $(2, 3, 3)$ what is the maximum rate of change of T?

(背面仍有題目,請繼續作答)

5. Find Laplace Transform of *y(t)* that satisfies the equation as follows:

$$d^2y/dt^2 + y = e^{-t}\int_0^t t\sin 2t\, dt \qquad \textbf{(12 \%)}$$

6. Find the solution of differential equation in power series, $x \neq 0$.

$$xd^2y/dx^2 + dy/dx - y = 0 \qquad \textbf{(15 \%)}$$

7. Find the Fourier Series of $f(x)$ that is defined as follows: **(8 %)**

$$f(x) = |x| \quad \text{where} \quad -1 \le x \le 1$$

8. The function f(z) = u(r, θ) + iv(r, θ) is given. Derive the Cauchy-Riemann equations in polar coordinates as **(15%)**

$$\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial\theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{r\partial\theta}$$

9. Solve the following problem (15%)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{for} \quad 0 < x < L, \quad t > 0$$

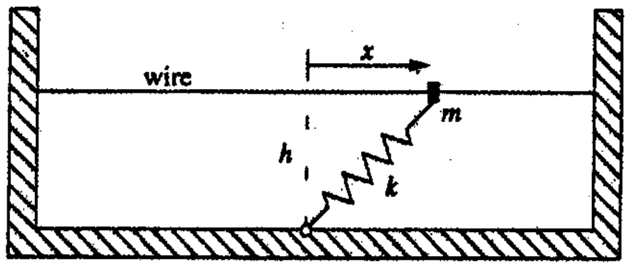
with

$$\frac{\partial u}{\partial x}(0,t) = -1, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

and

$$u(x, 0) = 0$$

1.



A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire (as sketched above). Suppose also that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

(a) Derive the *nonlinear* differential equation for the motion of the bead, and find *all* possible *equilibrium points* as functions of k, h, m, b, and L_0. (5%)

(b) The equation of motion derived in (a) can be linearized about $x = 0$ (which is obviously an equilibrium point) to yield

$$m\ddot{x} + b\dot{x} + k(1 - L_0/h)x = 0.$$

Suppose that $m = 0$. Integrate the above linearized equation of motion with an appropriate initial condition, and hence discuss the *stability* of the obvious equilibrium point $x = 0$. (5%)

(c) Now, suppose that $b = 0$ (but $m \neq 0$). Use *Laplace transform* to solve the linearized equation of motion in (b) with the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$. (10%)

2. (a) Complete the matrix $\mathbf{A}$ (fill in the two blank entries) so that $\mathbf{A}$ has eigenvectors $\mathbf{x}_1 = (3,1)^T$ and $\mathbf{x}_2 = (2,1)^T$:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ & \end{bmatrix} \quad (5\%).$$

(b) Find a matrix $\mathbf{B}$ with those same eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. Also calculate $\mathbf{B}^{10}$. (10%)

3. Given a two-dimensional vector field

$$\mathbf{V} = \frac{x^2 - y^2}{(x^2+y^2)^2}\mathbf{i} + \frac{2xy}{(x^2+y^2)^2}\mathbf{j}.$$

(a) Calculate by *direct integration* the *circulation* of $\mathbf{V}$ around the circular closed contour of radius 1 centered at the origin. To earn full credits, detail your calculations. (6%)

(b) Calculate the same circulation as in (a) by use of *Stokes's theorem*. Again, to earn full credits, detail your calculations. (9%)

（背面仍有題目,請繼續作答）

4. Show that substitution of $u = F(r,\theta)\,G(t)$ into the wave equation:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$$

leads to

$$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = ck$$

and

$$F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} + k^2 F = 0$$

(25%)

5. Prove the following series converges uniformly in the given region.

$$\sum_{n=0}^{\infty} \frac{z^n}{|z|^{2n} + 1}, \qquad 2 \le |z| \le 4$$

where z is the complex variable.

(25%)

本試題是否可以使用計算機： ☑可使用 ， □不可使用 （請命題老師勾選）

1. Please solve the following ordinary differential equation by the method of variation of parameters: (15%)

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=x\log x$$

2. Determine the coefficients in the representation (15%)

$$f(x)=\sum_{n=1}^{\infty}A_n\sin nx \qquad (0<x<\pi)$$

In the following cases:

(a) $f(x)=1$, (b) $f(x)=x$, (c) $f(x)=\begin{cases}1 & (x<\pi/2)\\ \frac{1}{2} & (x=\pi/2)\\ 0 & (x>\pi/2)\end{cases}$

3. Find the value of the following infinite integration, (20%)

$$\int_0^{\infty}\frac{\cosh ax}{\cosh x}dx \qquad \text{where } |a|<1$$

（背面仍有題目,請繼續作答）

本試題是否可以使用計算機： ☑可使用 ， □不可使用 （請命題老師勾選）

4. Find a). Fourier Transform,

b). Laplace Transform, and

c). Fourier Series

(assume period = $4a$, i.e. duty cycle= 50 %)

of f(x), where

$$\mathbf{f(x)} = \begin{cases} 1, & if \quad -a < x < a \\ 0, & otherwise \end{cases}$$

(21 %)

5. Find a unit normal vector $\vec{n}$ of the cone of revolution

$z^2 = 16\left(x^2 + 4\,y^2\right)$ at the point P: (1, 0, 4). (14 %)

6. A particle moves once counterclockwise about the rectangle with vertices (1, 1), (1, 7), (3, 1) and (3, 7) under the influence of the force $\vec{f}$, where

$\vec{f} = \left[-\cosh\left(4x^4\right) + xy\right]\vec{i} + \left(e^{-y} + x\right)\vec{j}$, where $\vec{i}$ and $\vec{j}$ are the unit vectors in X-axis and Y-axis respectively.

Calculate the work alone. (15%)

系所：機械工程學系甲、乙、丙、丁、戊組 科目：工程數學
本試題是否可以使用計算機： ☑可使用 ， □不可使用 （請命題老師勾選） 考試日期：0301，節次：3

1. Evaluate $I = \oint_C \cot(z)dz$, where C is the unit circle $|z| = 1$ traversed in a clockwise sense. (15%)

2. Let $f(x) = x^2/2$ for $-0 \le x \le \pi$. Find the Fourier series of f(x) and evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (15%)

3. Find the general solution of the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dt} + 2y = \delta(x-3)$.

(20%)

4. Solve $\dot{X} = AX$, where $X^T = [x_1 \quad x_2]$, $\dot{X} = dx/dt$,

$A = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$, the superscript "$T$" denotes transpose of a vector or matrix.

(15%)

5. Let $F(x, y, z) = (-y + z, x + yz, xyz)$. By applying Stokes' Theorem, compute the integral of Curl F over the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$, with outwards normals.

(15%)

6. Find the general solution of $y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x^2 + 1$, $x > 0$,

where $y' \equiv dy/dx$, $y'' \equiv d^2y/dx^2$.

(20%)