中山大學

光電所碩士班

91~97 學年度
工程數學考古題

國立中山大學九十一學年度碩士班招生考試試題

科目：工程數學（光電所碩士班）

1. Short questions (60 points) (10 points for each question)

(a) $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, then $\dfrac{3\Gamma\left(\frac{3}{4}\right)}{7\Gamma\left(\frac{11}{4}\right)} = ?$

(b) Evaluate the real and imaginary parts of $\cot\left(\frac{\pi}{3} - i\ln 3\right)$.

(c) $\int_0^\infty \dfrac{\cos\ mx}{x^2+1} dx = ?$, for $m > 0$.

(d) Explain *Gauss's theorem.*

(e) Explain *Stoke's theorem.*

(f) Express $\nabla^2 V = 0$ in spherical coordinates.

2. $\dfrac{d^2 y(x)}{d^2 x} - 2\dfrac{d^2 y(x)}{d^2 x} + 2y(x) = e^{-x}, y(0) = y\left(\frac{\pi}{2}\right) = 0,$ find y(x). (20 points)

3. Find the function $f(x,y)$ satisfying the Laplace equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0$$, for $x^2 + y^2 = a$, $a > 0$;

and the boundary condition $f(x,y) = x^3$ for $x^2 + y^2 = a$.
(20 points)

光電所工程數學

May 4 2003

1. Solve the following differential equation: (15%)

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0, \quad \text{boundary conditions:} \quad y(0)=1, y(1)=0.$$

2. Show that eigenvales (λ) of a real symmetric matrix **A** are real and that the eigenvectors ($\vec{x}$) are orthogonal to each other. (15%)

$$\mathrm{A}\vec{x}_i=\lambda_i\vec{x}_i, \quad \vec{x}_i^T\vec{x}_j=0, \quad i\neq j.$$

3. Using the fact that: (15 %)

$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$, perform the definite integral $\int_{-\infty}^{\infty} e^{-ax^2}\cos bx\,dx$.

4. Compute the complex refracted angle θ_2 under total reflection. This occurs when light enters from water (with high index n_1) to air (with low index $n_2=1$) with an incident angle θ_1 greater than the critical angle $\theta_c=\sin^{-1}(\frac{n_1}{n_2})$. You must express this angle both the real and imaginary parts in a closed form with hyperbolic functions (15%) (Hint: Apply the Snell law $n_1\sin\theta_1=n_2\sin\theta_2$)

5. a. State the Divergence Theorem and Stokes Theorem. (10%)
   b. Apply both theorems above show that: (10%)

$$\nabla\cdot(\nabla\times\vec{\mathrm{F}})\equiv 0$$

Here $\vec{\mathrm{F}}$ is a vector function of 3 spatial variables.

6. Find the first two eigen-functions for the 2-D Helmholtz equation satisfying Neumann boundary conditions. (Hint: apply the technique of separation of variables.) Please denote Bessel function as $J_n(z)$ and use $\chi_{n,i}$ as the i^{th} root of the derivative of the Bessel function: i.e. $J_n'(\chi_{n,i})=0$ (20%)

$$\left(\nabla^2+k^2\right)u(r,\theta)=0, \; r\in(0,1), \quad \theta\in(0,\frac{\pi}{2}),$$

$$\frac{\partial u(r,\theta)}{\partial n}=0 \quad \text{on the boundary, } \vec{n} \text{ is the normal vector.}$$

國立中山大學九十三學年度碩士班招生考試試題

科目：工程數學（光電所）

1. Please find the radius of convergence of the series: $\sum_{m=0}^{\infty}\frac{m}{3^m}(x-2)^m$. (8%)

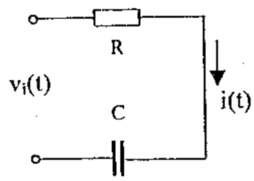
2. Please find the f(t) if its Laplace transformation is: $\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)$. (8%)

3. What are the eigenvalues and corresponding eigenvectors of the matrix $\underline{A}$? (8%)

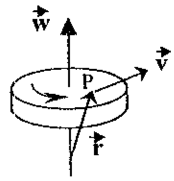
$$\underline{A}=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$$

4. Please find the unit normal vector of the cone of revolution $z^2 = 4(x^2 + y^2)$ at point (1, 0, 2). (8%)

5. By applying the Kirchhoff's law, please find the current i(t) in the RC circuit (given in the figure of question 5) if $v_i(t) = V_0\sin\omega t$. (8%)



Question 5

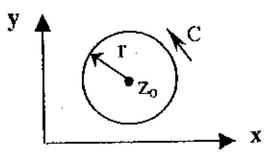


Question 6

6. What is the curl of the velocity field $\vec{v}$ of the rotating body shown in the figure of question 6. (10%)

7. Please show that the integral of z^{-3} around the unit circle is zero. (10%)

8. Please find $\int_C (z-z_0)^3\,dz$, where z_0 is a constant and the integration path C is shown in figure of question 8 (15%)



Question 8

9. Please solve the initial value problem consisting of equations:

$$\begin{aligned}\dot{y}_1 &= 5y_1 + 8y_2 + 1\\ \dot{y}_2 &= -6y_1 - 9y_2 + t\end{aligned}$$

, where $y_1(0) = 4$, and $y_2(0) = -3$. (15%)

10. What are the solutions of $f(x) = x^3 + x - 1 = 0$? (10%)

科目：工程數學【光電所碩士班】

1. $A=\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ (a) Is the matrix A Hermitian, skew-Hermitian or unitary?

(b) Please find its eigenvalues & eigenvectors. (10%)

2. Please find the inverse of the given linear transformation: (10%)

$x^*=19x+2y-9z$

$y^*=-4x-y+2z$

$z^*=-2x+z$

3. Please find the inverse Laplace transform of the function $\dfrac{3s+4}{s^2+4s+5}$ (10%)

4. Please evaluate the real integral: $\int_{-\infty}^{\infty}\dfrac{\cos x}{x^4+1}dx$ (10%)

5. Find the Fourier transform of the function f(x). (15%)

$f(x)=e^{-ax^2}$, where a > 0

6. Please solve the initial value problem:

$x^3y'''-3x^2y''+6xy'-6y=0, y(1)=2, y'(1)=1, y''(1)=-4 \quad (y^{(n)}\equiv\dfrac{d^ny}{dx^n})$ (15%)

7. Please evaluate $\oint_C \dfrac{2z^3+z^2+4}{z^4+4z^2}dz$, C the circle $|z-2|=4$, clockwise. (15%)

8. Please solve $x\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial t}=xt$, $u(x,0)=0$ if $x\geq 0$, $u(0,t)=0$ if $t\geq 0$. (15%)

國立中山大學95學年度碩士班招生考試試題

科目：工程數學【光電所碩士班】　　　　　　　　　　　共1頁第1頁

1. Solve $xy' = y + xy^2$ (10%)

2. Determine general solution to $y''' - y'' - 8y' + 12y = 7e^{2x}$ (15%)

3. (A) Show that $\int_0^\infty e^{-x^2}dx = \dfrac{\sqrt{\pi}}{2}$ (10%)　(B) Show that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$ (5%)

4. Diagonalizes A, $A = \begin{bmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{bmatrix}$ (15%)

5. Find the point (x, y, z) on the given plane $x - y + 2z = 4$, that is close to the point A(2, 0, -1), and the shortest distance. (15%)

6. Compute the line integral $\int_C \bar{F}(r)\cdot d\bar{r}$, where $\bar{F}(r) = y^2\bar{i} - x^2\bar{j}$
C is a straight-line segment from (0,0) to (1,2). (15%)

7. Solve the partial differential equation: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$,
I.C. $u(x, 0) = x$ and B.C. $\dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi,t) = 0$ (15%)

1. Please solve the initial value problem.

$(D^4+4D^3+8D^2+8D+4)y=0, y(0)=1, y'(0)=0, y''(0)=-2, y'''(0)=2$

$(D^{(n)}=\frac{d^n}{dx^n})$ (15%)

2. Please solve the differential equation. (15%)

$x(x-1)y''+(3x-1)y'+y=0$

3. Please evaluate the following integrals.

(a) $\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}$ $(a>1)$ (5%)

(b) $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$, taken counterclockwise around the circle $|z-2|=2$ (10%)

4. Find the center and the radius of convergence of the following power series.

(a) $\sum_{n=0}^{\infty}(\frac{4-2i}{1+5i})^n z^n$ (5%) (b) $3^2z^2+z^3+3^4z^4+z^5+3^6z^6+z^7+\ldots$ (5%)

5. Find the Taylor series of the following function with center $z_0=1$. (15%)

$f(z)=\frac{2z^2+9z+5}{z^3+z^2-8z-12}$

6. In an experiment to monitor two calls, the Probability Mass Function (PMF) of N the number of voice calls, is

$P_N(n)=\begin{cases}0.1 & n=0\\ 0.4 & n=1\\ 0.5 & n=2\\ 0 & otherwise\end{cases}$ Please find

(a) The mean square value $E[N^2]$ (5%) (b) The standard deviation σ_N (5%)

7. Fit a straight line to the given points (x,y) by the method of least squares. (Ohm's law : U=Ri) Estimate R from the least squares line that fits the following data. (i,U)=(3.0, 162), (5.0, 255), (7.0, 360), (10.0, 495). (10%)

8. Solve the following linear system by Gauss elimination. (10%)

$2x_1+5x_2+7x_3=25$

1. Solve the initial value problem:
   $y'' + y = 0.001\, x^2$, $y(0) = 0$, $y'(0) = 1.5$ (15 %)

2. Find the value of $\int_C F(r)dr = \int_a^b F(r(t))r'(t)dt$

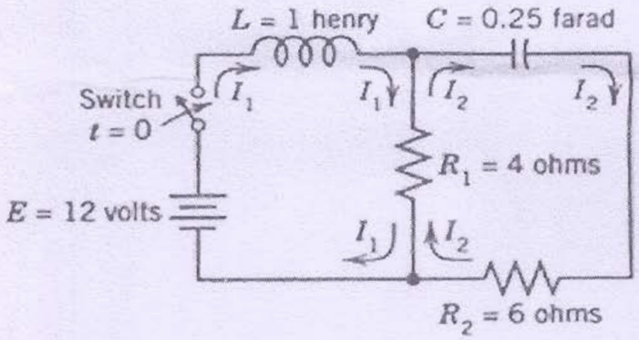
   when $F(r) = [z, x, y] = z\,\hat{i} + x\,\hat{j} + y\,\hat{k}$ and C is a helix:

   $r(t) = [\cos t, \sin t, 3t] = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$ $(0 \leqq t \leqq 2\pi)$

   (20 %)

3. Find the Fourier series of the function:
   $f(x) = x + \pi$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ (20 %)

4. Find the currents $I_1(t)$ and $I_2(t)$ in the figure below. Assume all currents and charges to be zero at t=0, the instant when the switch is closed. (15 %)



5. Find the temperature $u(x, t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100 \sin(\pi x/80)$ °C and the ends are kept at 0 °C. How long will it take for the maximum temperature in the bar to drop to 50 °C? Assume physical data for copper: density 10 g/cm^2, specific heat 0.1 cal/g.°C, and thermal conductivity 1 cal/cm.s.°C. (15 %)

6. Find an upper bond for the absolute value of the integral:

   $\int_C z^2 dz$, C the straight-line segment from 0 to $1+i$ (15 %)

中山大學

海洋環境與工程學系碩士班

91~97 學年度
工程數學考古題

國立中山大學九十一學年度碩士班招生考試試題

科目： 工程數學【海洋環境及工程學系碩士班】(甲組必考)

1. [二階線性微分方程式] (每小題各 10 分；共 20 分)
   (a) 以未定係數法求解下列方程式之初值問題：
   $y''+4y'+y=2\cos x+3\sin x; y(0)=1, y'(0)=0$。
   (b) 以參數變化法求解下列方程式之通解：
   $y''-2y'+y=x^{1/2}e^{x}$。

2. [線性方程式組、矩陣] (每小題各 10 分；共 20 分)
   (a) 以數學式表示一無限多解、僅有一解及一無解的線性方程式組之簡例。
   (b) 求下列矩陣之特徵值及特徵向量 $\begin{bmatrix}1 & 0\\ 0 & -3\end{bmatrix}$。

3. [向量之旋度、散度] (每小題各 10 分；共 20 分)
   (a) 已知一穩定流流體之速度向量 $\mathbf{v}=-y^2\mathbf{i}+2\mathbf{j}$,
   證明該流體具不可壓縮性並試求流體內任一質點之運動路徑方程式。
   (b) 向量場 $\vec{F}=2z^2-y^2-x^2$ 通過一矩形體
   在 $0\le x\le 1,\ 0\le y\le 2,\ 0\le z\le 4$ 之表面 S；
   證明 $\vec{F}$ 為一諧和函數(harmonic function)及 $\iint_S \frac{\partial \vec{F}}{\partial n}dA=0$。

4. [偏微分方程式] (每小題各 10 分；共 20 分)
   (a) 線性偏微分方程式 $Au_{xx}+2Bu_{xy}+Cu_{yy}=F(x,y,u,u_x,u_y)$ 之類型有三。
   試列出各類型之名稱、代表條件(A、B、C 之組合)及常見的代表數學式
   (如 Laplace、熱傳導及波動方程式)。
   (b) 試證明 $u=\dfrac{2xy}{(x^2+y^2)^2}$ 為 Laplace 方程式之一解。

5. [留數積分法] (每小題各 10 分；共 20 分)
   (a) 以留數積分法求 $\oint_C \dfrac{e^{-z^2}}{\sin 4z}dz$。
   (b) 以留數積分法求 $\int_0^{\infty}\dfrac{1+x^2}{1+x^4}dx$。

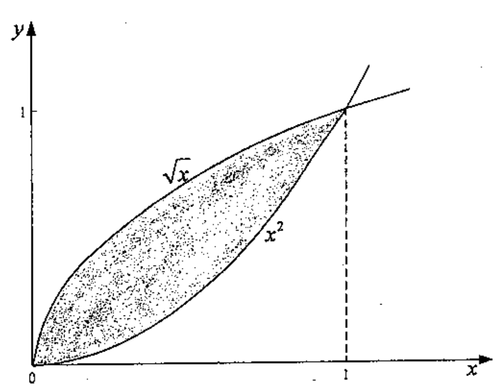
1. About Basic Concepts. (15%)
   (a) What is Calculus? (5%)
   (b) Describe its relationship with Mathematical Modeling. (5%)
   (c) Give one specific example of using Calculus in the model formulation. (5%)

2. About Limit. (10%)
   (a) Evaluate $\lim_{x\to 1} f(x)$ *where* $f(x)=\begin{cases}\dfrac{x^2-x}{x-1} & \text{if } x<1\\ \sqrt{1-x} & \text{if } x\geq 1\end{cases}$ (5%)
   (b) Find the intervals on which the function $f(x)=\left|x^2-4\right|$ is continuous. (5%)

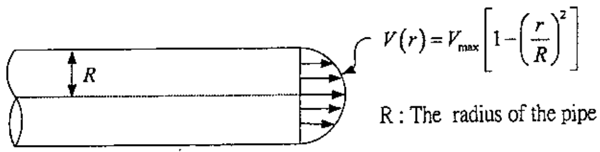
3. About Derivative and Integral. (35%)
   (a) Evaluate dy/dx given the condition that y is a differentiable function of x that satisfies $\sin(x^2+y)=y^2(3x+1)$. (10%)
   (b) Find $\dfrac{d\sin^{-1}\left(x/a\right)}{dx}$ and $\dfrac{d(x+1)^x}{dx}$. (8%)
   (c) Evaluate $\int_1^2 x^2dx$ by using the definition of Definite Integral (the limit of a Riemann sum). (10%)
   (d) Find $\int\dfrac{\ln\sqrt{x}}{\sqrt{x}}dx$ (7%)

4. About Sequences and Series. (10%)
   (a) Show if the sequence $n^{1/n}$ is convergent or divergent? (5%)
   (b) Test the series $\sum_{x=1}^{\infty}\dfrac{(-1)^{x+1}\ln x}{x}$ for convergence. (5%)

5. Find the magnitude of the shadow area and its centroid $(\bar{x},\bar{y})$. (15%)



6. A pipe flow it's maximum velocity (V_{max}) locates at the center of the pipe, and it's velocity profile as show in the diagram. Find the average velocity $\overline{V}$ of the pipe in terms of V_{max}. (15%)



科目：工程數學（含高等工程數學）【海洋環境及工程學系碩士班】（甲組） 共 1 頁 第 1 頁

1、【**Ordinary differential equation**】**10%**

Evaluate a general solution for the following second-order differential equation:

$$y''-2y'+y=x^{\frac{3}{2}}e^{x}$$

2、【**Ordinary differential equation**】**25%**

Solve the following initial value problems in second-order differential equation:

1). $(D^2+2D+10)y=10x^2+4x+2,\quad y(0)=1,\quad y'(0)=-1$

2). $(D^2+4D+1)y=2\cos x+3\sin x,\quad y(0)=0,\quad y'(0)=0$

3、【**Partial differential equation**】**20%**

Linear partial differential equations, $Au_{xx}+2Bu_{xy}+Cu_{yy}=F(x,y,u,u_x,u_y)$ can be classified into one of the three types: elliptic, parabolic or hyperbolic, depending on the condition of B^2-4AC.

1). Indicate how the condition of B^2-4AC is linked to each of these three types and also provide a typical mathematical equation for each of them.

2). Prove $u=2xy/(x^2+y^2)^2$ is a solution to the Laplace equation.

4、【**Laplace transform**】**15%**

Solve the following initial value problem by the Laplace transform

$y''-5y'+6y=r(t)$, where $r(t)=4e^t$, for $0<t<2$, and $r(t)=0$, for $t>2$;

with initial conditions $y(0)=1$, and $y'(0)=-2$.

5、【**Fourier analysis**】**10%**

Find the Fourier series of $f(x)=e^{2x}$ on [0, 1].

6、【**Vector analysis**】**10%**

Evaluate the surface integral $\iint_S (curl\mathbf{F})\bullet\mathbf{n}dA$, given

$\mathbf{F}=[-e^y,e^z,e^x]$, S the square $0\le x\le 1,\quad 0\le y\le 1,\quad z=x+y$

7、【**Residue integration**】**10%**

Evaluate the following integral $\oint_C \frac{e^{-z^2}}{\sin 4z}dz$

1. What is the purpose of Calculus? Describe the contents that you have studied in the college? (10%)

2. Evaluate the following limit values (20%)

 (a) $\lim\limits_{x\to 0}\dfrac{\sin(ax)}{\sin(bx)} = ?$ (5%)

 (b) $\lim\limits_{x\to 0} x^{x} = ?$ (5%)

 (c) $\lim\limits_{x\to 0}(x\sin(1/x)) = ?$ (5%)

 (d) $\lim\limits_{x\to 2}\dfrac{3x-5}{x-2} = ?$ (5%)

3. Evaluate the following derivatives (20%)

 (a) $\dfrac{d}{dx}\left(e^{-3x}\sin(x)\right) = ?$ (5%)

 (b) $\dfrac{d}{dx}\left(\sin^{-1}(x^2)\right) = ?$ (5%)

 (c) $\dfrac{d}{dx}(1+x)^{2x} = ?$ (5%)

 (d) $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{x^2-3x}\tan(t)dt = ?$ (5%)

4. Evaluate the following integrals (20%)

 (a) $\displaystyle\int\frac{x^2}{(x^3-2)^2}dx = ?$ (5%)

 (b) $\displaystyle\int 3^{5-x}dx = ?$ (5%)

 (c) $\displaystyle\int_0^{+\infty} xe^{-2x}dx = ?$ (5%)

 (d) $\displaystyle\int_0^3 (x-2)^{-1}dx = ?$ (5%)

5. An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide concentration in the air is described by the formula

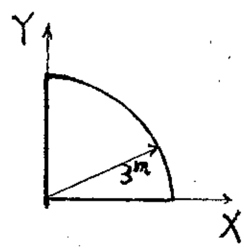
$$C(p)=\sqrt{0.5p^2+17}$$

Parts per million when the population is p thousand. It is estimated that t years from now, the population of the community will be

$$p(t)=3.1+0.1t^2$$

What will the change rate of carbon monoxide concentration be after 3 years from now? (15%)

6. A quarter circular plate with radius 3m is shown in the figure. Use integral method to find its area, and find the Centroid $(\bar{X},\bar{Y})$ of the plate. (15%)



1、【**Ordinary Differential Equations**】 **30%**

(a) Find the general solution of $(1+2e^{x/y})\ dx + 2e^{x/y}(1-x/y)\ dy = 0$

(b) Find a second solution of the following 2nd order linear differential equation using the given y_1:

$(1-x^2)y'' - 2xy' + 2y = 0, \quad y_1 = x$

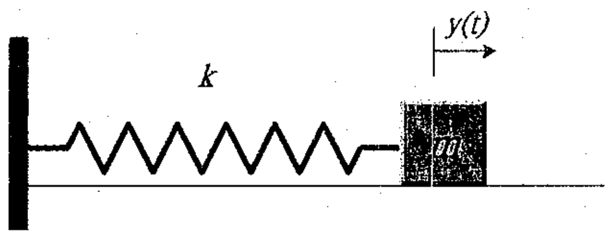
2、【**Linear Algebra**】 **20%**

(a) A linear system $\mathbf{Ax} = \mathbf{b}$ of m equations and n variables. State under what conditions the linear system will have solution(s), a unique solution, and infinitely many solutions.

(b) Given a matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, find its orthonormal basis of eigenvectors for R^3.

3、【**Modeling Spring Motion**】 **10%**

Suppose that a body of mass m slides without friction on a horizontal surface as shown in the following figure. The body is attached to a spring with spring constant k (the damping force of the spring is excluded), and is also subject to viscous air resistance with coefficient γ (the higher speed, the more air resistance). Formulate a differential equation to simulate this spring motion system, and briefly describe how to solve it.



4、【**Laplace transform and solution by undetermined coefficients**】 **20%**

(a) Solve the following linear differential equation by Laplace transform

$y''+2y'+y = e^{-t}$, for initial conditions $y(0) = -1$ and $y'(0) = 1$.

(b) Solve the following linear differential equation by solution of undetermined coefficients

$y''+2y'+y = e^{-t}$, for initial conditions $y(0) = -1$ and $y'(0) = 1$.

5、【**Fourier analysis**】 **10%**

Find the Fourier series for a periodic square wave given by the function

$$f(x) = \begin{cases} 0, if -2 < x < -1 \\ k, if -1 < x < 1; p = 2L = 4, L = 2 \\ 0, if 1 < x < 2 \end{cases}$$

6、【**Vector analysis – divergence – conservation of mass in fluid flow**】**10%**

Consider the flow through a rectangular box R with dimensions $\Delta x, \Delta y, \Delta z$ parallel to the Cartesian coordinate axes. The velocity vector of the fluid particle is $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. From conserving the flux of fluid mass entering and leaving the boundary faces of the box per unit time, derive the continuity equation of a compressible fluid flow, i.e.,

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0,$$

where the divergence $div(\rho v) = \dfrac{\rho \Delta v_1}{\Delta x} + \dfrac{\rho \Delta v_2}{\Delta y} + \dfrac{\rho \Delta v_3}{\Delta z}$.

科目：工程數學【海工系碩士班甲組】

(1) (20%) 求解下列微分方程式

(a) $y' + \frac{1}{2}y = y^3, y(0) = 1$ (10%)

(b) $y'' + 2y' - 3y = 8e^{-t} + \delta(t - \frac{1}{2}), y(0) = 3, y'(0) = -5$ ， $\delta()$ 是單位脈衝函數（unit impulse function）. (10%)

(2) (10%) 已知矩陣 $\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 4 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

(a) 求矩陣 **A** 的 rank. (5%)

(b) 假設有一個線性系統 $\mathbf{Ax} = \mathbf{0}$, **A** 是本題所給的矩陣. 求矩陣 **A** 的 null space 以及它的 nullity. (5%)

(3) (10%) The velocity vector **v** of an incompressible fluid rotating in a cylindrical vessel is of the form $\mathbf{v} = \mathbf{w} \times \mathbf{r}$, where **w** is the rotation vector and **r** is the position vector. Show that there is no divergence in this flow field.

(4) (10%) 求面積分 $\iint_S \mathbf{F} \bullet \mathbf{n}\, dA$， $\mathbf{F} = [x^3, y^3, z^3]$ ，S 為球面 $x^2 + y^2 + z^2 = 4$.

(5) (10%) 求 $\sin x, 0 < x < \pi$ 之 Fourier 級數.

(6) (15%) 求 $\sum_{n=0}^{\infty} a^n \cos n\theta$ 之級數和.

(7) (15%) 導出 Fourier integral.

(8) (10%) 由分離變數法解 $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$ 之通解

科目：微積分【海工系碩士班丙組選考】

1. (20%) About Continuity.
   a) The temperature on a specific day at a given location is considered as a function of time. Is this special function a continuous function? Explain your answer. (7%)
   b) The charge for a taxi ride is considered as a function of mileage. Is this special function a continuous function? Explain your answer. (7%)
   c) Given a real function $f(x)=\begin{cases}\dfrac{\sin ax}{x} & if\ x<0\\ 5 & if\ x=0\\ x+b & if\ x>0\end{cases}$, find constants a and b so that $f(x)$ is continuous. (6%)

2. (15%) About Limit.
   a) Determine $\lim\limits_{x\to 0}\cos\left(\dfrac{1}{x}\right)$ (7%)
   b) Determine $\lim\limits_{x\to 0}\dfrac{e^{-4x}-1}{e^{-2x}+e^{-x}-2}$ (8%)

3. (20%) About Differentiation.
   a) Find the derivative $\dfrac{dy}{dx}$ in terms of x alone using $y=\dfrac{e^{2x}(2x-1)^6}{(x^3+5)^2(4-7x)}$. (10%)
   b) In a healthy person of height x inches, the average pulse rate in beats per minute is modeled by the formula $P(x)=596/\sqrt{x}$, $30\le x\le 100$. Estimate the change in pulse rate that corresponding to a height change from 59 to 60 inches. (10%)

4. (30%) About Integration.
   a) Evaluate $\int[\sin x\ \ln(2+\cos x)]dx$ (15%)
   b) Evaluate $\int x^3\sqrt{x^2-1}\,dx$ (15%)

5. (15%) About Infinite Series.
   a) Test the series $\sum\limits_{k=1}^{\infty}\dfrac{2k^3+k+1}{k^3+k^2+1}$ for convergence or divergence. (7%)
   b) Find the interval of convergence for the power series $\sum\limits_{k=1}^{\infty}\dfrac{(x+1)^k}{3^k}$. (8%)

國立中山大學95學年度碩士班招生考試試題

科目：工程數學【海工系碩士班甲組】

1. (20%) 求解下列常微分方程式.

   (a) (10%) $x^3y' + x^2y = 2y^{-4/3}$

   (b) (10%) $y'' + y' - 2y = -6\sin 2x - 18\cos 2x, \quad y(0) = 2, y'(0) = 2$

2. (10%) 利用**兩種**不同的方法找出拉普拉斯(Laplace)的逆轉換 $\mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right]$

3. (10%) 回答下列有關矩陣的問題.

   (a) (2%) 找出矩陣 $\mathbf{A}\begin{bmatrix} 5 & 0 & -15 \\ -3 & -4 & 9 \\ 5 & 0 & -15 \end{bmatrix}$ 的秩(rank).

   (b) (6%) 找出矩陣 $\mathbf{A}$ 的特徵值(eigenvalues) 與特徵向量(eigenvectors).

   (c) (2%) 你能將矩陣 $\mathbf{A}$ 對角化嗎 (diagonalize the matrix $\mathbf{A}$)? 說明你的理由?

4. (10%) 已知一向量函數 $\mathbf{F} = z\,\mathbf{i} - xz\,\mathbf{j} + y\,\mathbf{k}$,在給定的曲面(surface)$S$ 上 $S: x^2 + 9y^2 + 4z^2 = 36$

   , $x \geq 0, y \geq 0, z \geq 0$,請計算其通量積分(flux integral) $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA$

5. (5%) 已知 $\frac{1}{a^2+x^2}$ 的 Fourier Transform 為 $-\sqrt{\frac{\pi}{2}}\frac{e^{a|w|}}{a}$,求 $\frac{x}{(a^2+x^2)^2}$, $\mathrm{Re}(a) < 0$ 的 Fourier

   Transform.

6. (15%) 有關 PDE

   (a) (10%) 推導 PDE

   一弦之縱向張力為常數 T,密度為 $\rho(x)$,單位長度橫向受力為 $f(x)$,請推導其控制方程式;弦之橫向位移請以 y 表示。

   (b) (5%) 以上方程是橢圓/拋物線/雙曲線中的哪一類方程?

7. (20%) Fourier Series 與 Parseval Relation

   (a) (10%) 函數 $f = x, -\pi < x < \pi; f(x+2\pi) = f(x)$,求 f 的 Fourier Series。

   (b) (10%) 求級數和 $\sum_{m=1}^{\infty}\frac{1}{m^2}$

8. (10%) 請用直接積分的方式求複變積分 $\oint_C \frac{1}{z^2}dz$,其中 C 是以原點為圓心的單位圓。

[redacted]

1. (20%) Find the derivative for the following given functions with respect to x or θ or λ:

   (a) $f(x)=\dfrac{x+1}{\sqrt{x}}$; (b) $f(\theta)=\dfrac{\theta}{1-\sin\theta}$; (c) $f(x)=\sqrt[3]{(x^2-1)^2}$; (d) $f(\lambda)=\ln\dfrac{e^{-\lambda}\lambda^{x}}{x!}$.

2. (10%) Given (a) $x^2+y^2=25$, find $\dfrac{d^2y}{dx^2}$; (b) $z=ye^{2x}+x\ln y^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial^2 z}{\partial x\partial y}$.

3. (5%) Find the limits for:

   (a) $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x-1}$; (b) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$.

4. (5%) If the bacteria growth in a culture can be expressed as a function in time t, such as $P(t)=1000\left(1+\dfrac{4t}{50+t^2}\right)$, where the initial number of bacteria was 1000 and t is for the time in hours. Find the rate of population growth for the bacteria when $t=2$ hours.

5. (10%) Find the relative extrema for function $f(x)=-3x^5+5x^3$, using first and second order derivatives to assist the classification of either a maximum or minimum.

6. (40%) Find the integral for each of the following functions:

   (a) (15%) $\displaystyle\int\frac{dx}{x^{1/3}+x^{1/2}}$.

   (b) (10%) $\displaystyle\int[\sin x\cdot\ln(2+\cos x)]\,dx$.

   (c) (15%) $\displaystyle\int\sqrt{16x-2x^2-23}\,dx$.

7. (10%) About Series

   (a) (5%) Test the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{\sin k}{2^k}\right|$ for convergence or divergence.

   (b) (5%) Find the interval of convergence for the power series $\displaystyle\sum_{k=1}^{\infty}\frac{2^k x^k}{k}$.

國立中山大學96學年度碩士班招生考試試題

科目：工程數學【海工系碩士班甲組】

(1) **【Ordinary Differential Equations】** (24%)

(a) Find the general solution of $y'(\sinh 3y - 2xy) = y^2$. (8%)

(b) Solve $y'' + 2y' + 2y = 4e^{-x}\sec^3 x$. (8%)

(c) Solve $xy'' + (1-x)y' + ny = 0$ by Laplace Transformation, where n is a non-negative integer. (8%)

(2) **【Linear Algebra】** (10%)

Reduce $2x_1^2 + 12x_1x_2 - 7x_2^2 = 10$ to principal axes, and express $[x_1\ \ x_2]^T$ in terms of new variables. (10%)

(3) **【Vector Calculus】** (16%)

(a) Given $f = xy - yz$, $\mathbf{v} = [2y, 2z, 4x+z]$, $\mathbf{w} = [3z^2, 2x^2 - y^2, y^2]$, find $D_{\mathbf{v}} f$ (directional directive of f in the direction of $\mathbf{v}$) at $(2, 3, 1)$ and $[(curl\,\mathbf{v}) \times \mathbf{w}] \bullet \mathbf{w}$. (8%)

(b) Find the work done by a force $\mathbf{F} = [x^2, y^2, y^2 x]$ along the curve C: the helix $\mathbf{r} = [\cos t, \sin t, 3t]$, $0 \le t \le \pi/2$. (8%)

(4) **【Partial Differentiation Equations】** (24%)

Linear partial differential equations, $Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$ can be classified into one of the three types: elliptic, parabolic or hyperbolic, depending on the condition of $B^2 - 4AC$.

(a) Indicate how the condition of $B^2 - 4AC$ is linked to each of these three types and also provide a typical mathematical equation for each of them. (15%)

(b) Prove $u = 2xy/(x^2 + y^2)^2$ is a solution to the Laplace equation. (9%)

(5) **【Fourier Analysis】** (10%)

Find the Fourier series for $f(x) = x$ within the range of $0 < x < 2\pi$. (10%)

(6) **【Solution by Laplace Transform and Undetermined Coefficients】** (16%)

Solve the following linear differential equation $y''+2y'+y = e^{-t}$, by

(a) Laplace transform for initial conditions $y(0) = -1$ and $y'(0) = 1$. and (8%)

(b) Solution of undetermined coefficients for initial conditions $y(0) = -1$ and $y'(0) = 1$. (8%)

國立中山大學96學年度碩士班招生考試試題

科目：微積分【海工系碩士班丙組選考】　　　　共1頁第1頁

1. (20%) Find the first derivative for the following given functions with respect to x:

(a) $f(x)=\dfrac{4+2x}{\sqrt{x^3}}$; (b) $f(x)=x^2e^{-x}$; (c) $f(x)=5x\sqrt{x^2+1}$; (d). $x\ln y-y\ln x=8$.

2. (10%) Given (a) $x^2+y^2=3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$;

(b) $z=ye^{2x}+x\ln y^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial^2 z}{\partial x\partial y}$.

3. (5%) Find the limits for:

(a) $\lim\limits_{x\to 1}\dfrac{3x\ -3}{x^2-1}$; (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{x}-1}{x-1}$.

4. (5%) A circular cylindrical container is 20 cm high with 6 cm in inner diameter. Initially, the container is filled of water to its full height and is placed vertically. If a small hole is punctured at the bottom of the container and allows water to discharge by gravity. Estimate the rate of decrease in water surface level for the initial rate of discharge at 12 cm^3 per second.

5. (10%) Find the relative extrema for function $f(x)=2x\ -3x^{2/3}$, using first and second order derivatives to assist the classification of either a maximum or minimum.

6. (40%) Find the integral for each of the given functions:

(a) $\displaystyle\int\frac{dx}{x^{1/3}+x^{1/2}}$.

(b) $\displaystyle\int\frac{t+1}{t^{1/2}}dt$

(c) $\displaystyle\int[\sin x\cdot\ln(2+\cos x)]\,dx$.

(d) $\displaystyle\int_1^e\frac{3x^2+4}{x}dx$

7. (10%) Questions in power series:

(a) Test whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{n-i}{3n+2i}$ is in convergence or divergence.

(b) Find the Fourier series for $f(x)=x$ within the range of $0<x<2\pi$.

1. 【**Ordinary Differential Equations**】 **(20%)**

(a) Use **two** different methods to solve the ODE: $-2x\sin(x^2)\,dx+\dfrac{\cos(x^2)}{y}\,dy=0$.

(b) Use Laplace Transformation to the IVP (Initial Value Problem) $y''+5y'+6y=\delta(t-\frac{1}{2}\pi)+u(t-\pi)\cos t$, $y(0)=0$, $y'(0)=0$, where δ () is an unit impulse function and $u()$ is an unit step function.

2. 【**Linear Algebra**】 **(10%)**

(a) Given a homogeneous linear system, whose number of variables is 54 and the number of equations is 30, will this linear system have non-trivial (i.e. non-zero) solutions and what's the dimension of the solution space (you may make some assumption here)? Explain your answer?

(b) Given $\hat{\mathbf{A}}=\mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}$, where $\mathbf{A}=\begin{bmatrix}7&0&3\\2&1&1\\2&0&2\end{bmatrix}$, $\mathbf{T}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, Find the eigenvalues for both $\mathbf{A}$ and $\hat{\mathbf{A}}$.

3. 【**Vector Calculus**】 **(20%)**

(a) Given a curve $C: x^2+y^2=4$, $z=6\,arc\tan(y/x)$, represent the curve by the parametric form. Find the length along the curve from (2, 0, 0) to (2, 0, 24π).

(b) Evaluate the surface integral $\iint_S curl\,\mathbf{F}\bullet\mathbf{n}\,dA$, where $\mathbf{F}=[y^3,-x^3,0]$, $S: x^2+y^2\le 1$, $z=0$. Then verify your answer by Stokes's theorem.

4. 【**Partial differential equation**】 **20%**

Linear partial differential equations, $Au_{xx}+2Bu_{xy}+Cu_{yy}=F(x,y,u,u_x,u_y)$ can be classified into one of the three types: elliptic, parabolic or hyperbolic, depending on the condition of B^2-4AC.

(a) Indicate how the condition of B^2-4AC is linked to each of these three types and also provide a typical mathematical equation for each of them. (15%)

(b) Prove $u=2xy/(x^2+y^2)^2$ is a solution to the Laplace equation. (5%)

5. 【**Fourier analysis**】 **10%**

Find the Fourier series for a periodic square wave given by the function:

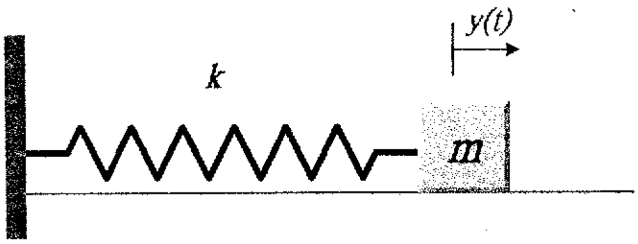
$$f(x)=\begin{cases}0, if -2<x<-1\\ k, if -1<x<1; p=2L=4, L=2\\ 0, if 1<x<2\end{cases}$$

6. 【**Residue integration**】 **10%**

Evaluate the following integral $\oint_C \frac{e^{-z^2}}{\sin 4z} dz$

7. 【**Modeling spring motion**】 **10%**

Suppose that a body of mass m slides without friction on a horizontal surface as shown in the following figure. The body is attached to a spring with spring constant k (the damping force of the spring is excluded), and is also subject to viscous air resistance with coefficient γ (the higher speed, the more air resistance). Formulate a differential equation to simulate this spring motion system, and briefly describe how to solve it.



Part 1: Differentiation and limits (50%)

1. (20%) Find the derivative for each of the following given functions with respect to x or θ or λ:

(a) $f(x)=\dfrac{x+1}{\sqrt{x}}$; (b) $f(x)=\sqrt[3]{(x^2-1)^2}$;

(c) $f(\theta)=\dfrac{\theta}{1-\sin\theta}$; (d) $f(\lambda)=\ln\dfrac{e^{-\lambda}\lambda^y}{y!}$.

2. (10%) (a) Given $x^2+y^2=25$, find $\dfrac{d^2y}{dx^2}$;

(b) Given $z=ye^{2x}+x\ln y^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial^2 z}{\partial x\partial y}$.

3. (10%) Water discharges into a large conical tank with its top open. The radius of the top is 5 m and vertical height of the tank is 10 m. If water is running at the constant rate of 2 m^3 per minute, how fast is the water level rising when the water is 6 m deep from its bottom tip? 【Hint: Volume of a whole conical shape = $\pi r^2h/3$】

4. (10%) Find the limits for:

(a) $\lim\limits_{x\to\infty}\dfrac{2x-1}{\sqrt{3x^2+x+1}}$; (b) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$

Part 2: Integration (50%)

5. (10%) $\displaystyle\int_0^4\frac{x+2}{\sqrt{2x+1}}dx$

6. (10%) $\displaystyle\int\frac{\cos 2x}{\sin^3 2x}dx$

7. (10%) $\displaystyle\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

8. (10%) $\displaystyle\int x^2e^x dx$

9. (10%) Find the area of the region bounded by the graph of $y=\dfrac{2x}{\sqrt{x^2+9}}$, with $y=0$, $x=0$, and $x=4$.

中山大學

電機系碩士班

91~97 學年度

工程數學考古題

1. (10%) We would like to evaluate an integral involving the derivative of the Dirac δ-function.

   (a) Consider $I=\int_{-\infty}^{\infty}\cos t\,\delta'(t-\frac{\pi}{2})\,dt$. Let $u(t)=\delta(t-\frac{\pi}{2})$. Then $I=\int_{t=-\infty}^{\infty}\cos t\,du$.

   Make use of integration by part to obtain the answer.

   (b) Find the a general formula for $\int_{-\infty}^{\infty}x(t)\delta'(t-t_0)\,dt$.

2. (15%) Let $x(t)$ be a rectangular pulse defined by $x(t)=1$, $|t|<1/2$ and $x(t)=0$, otherwise. The corresponding Fourier transform is denoted as $X(j\omega)$, i.e.,

   $X(j\omega)=\int_{-\infty}^{\infty}x(t)\exp(-j\omega t)\,dt$. Calculate

   (a) $X(j\omega)$, (b) $\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}d\omega$, (c) $\int_{-\infty}^{\infty}\frac{\sin\omega\cos(2\omega)}{\omega}d\omega$

3. (10%) Let $\vec{F}=\vec{a}_x 2xy+\vec{a}_y x^2+\vec{a}_z(z-1)$. Evaluate the line integral $\int_{(0,0,0)}^{(1,1,0)}\vec{F}\cdot d\vec{\ell}$ along a parabola $y=x^2$ on the xy plane.

4. (14%)

   (a) (6%) Let $A\in\mathbb{R}^{m\times n}$. Show that: $N(A^TA)=N(A)$.

   (b) (8%) Let U and V be subspaces of a vector space W and suppose $W=U+V$. Show that $U\cap V=\{0\}$ if and only if for any $w\in W$, there exist a unique $u\in U$ and a unique $v\in V$ such that $w=u+v$.

5. (16%) Let $A\in\mathbb{R}^{n\times n}$ have no eigenvalues being 1 and -1.

   (a) (8%) Show that $A^T=-A$ if and only if e^{At} is an orthogonal matrix for all t.

   (b) (8%) Show that $A^T=-A$ if and only if $(I-A)(I+A)^{-1}$ is an orthogonal matrix.

6. (15%) Find the general solution of the following differential equation:

   $6xy\,dx+(4y+9x^2)\,dy=0$

7. (20%) Use the Laplace transform to solve the following initial value problem:

   $y''+2y'+2y=\sin t$, $y(0)=1$, $y'(0)=1$.

1. (15%) Let $x(t)$ be a rectangular pulse defined by $x(t)=1$, $|t|<1/2$ and $x(t)=0$, otherwise. The corresponding Fourier transform is denoted as $X(j\omega)$, i.e.,

$X(j\omega)=\int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\,dt$. Calculate

(a) $X(j\omega)$, (b) $\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}d\omega$, (c) $\int_{-\infty}^{\infty}\frac{\sin\omega\cos(2\omega)}{\omega}d\omega$

2. (10%) Let $\vec{F}=\vec{a}_x 2xy+\vec{a}_y x^2+\vec{a}_z(z-1)$. Evaluate the line integral $\int_{(0,0,0)}^{(1,1,0)}\vec{F}\cdot d\vec{\ell}$ along a parabola $y=x^2$ on the xy plane.

3. (16%)

(a) (8%) Let $A\in\mathbb{R}^{m\times n}$. Show that: $N(A^TA)=N(A)$.

(b) (8%) Let U and V be subspaces of a vector space W and suppose $W=U+V$. Show that $U\cap V=\{0\}$ if and only if for any $w\in W$, there exist a unique $u\in U$ and a unique $v\in V$ such that $w=u+v$.

4. (24%)

(a) (7%) Let $A\in\mathbb{R}^{n\times n}$. Show that A is skew symmetric if and only if e^{At} is an orthogonal matrix for all t.

(b) (6%) Consider the dynamic system

$$\dot{x}(t)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}x(t)\quad\text{with}\quad x(0)=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\neq\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Use *similar transformation technique* to *diagonalize* the system matrix, then solve $x(t)$.

(c) (6%) (i) Find the size of $x(t)$ in the normed space $(\mathbb{R}^2,\|\cdot\|_2)$ for all $t\geq 0$.

(ii) Find all $t\geq 0$ such that $x(t)$ is orthogonal to $x(0)$.

(d) (5%) Explain the results of (c) from exploiting the property proved in (a).

5. (15%) Find the general solution of the following differential equation:

$$6xy\,dx+(4y+9x^2)\,dy=0$$

6. (20%) Use the Laplace transform to solve the following initial value problem:

$$y''+2y'+2y=\sin t,\quad y(0)=1,\quad y'(0)=1.$$

1. (20%) A periodic function with a period 2 is defined by $x(t)=|t-1|$, $0 \le t \le 2$. Expand $x(t)$ in Fourier series

$$x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}=a_0+\sum_{k=1}^{\infty}[a_k\cos(k\omega_0 t)+b_k\sin(k\omega_0 t)],$$ where ω_0 is the fundamental angular frequency. Determine

(a) $X[k]$ (5pts),

(b) a_k and b_k, $k=1,2,3,\cdots$ (10pts),

(c) $\sum_{k=1}^{\infty} a_k \cos(\frac{1}{2}k\omega_0)$. (5pts)

2. (15%) In the following $\vec{c}$ is an arbitrary constant vector.

(a) Show that if $\vec{c}\cdot\vec{A}=\vec{c}\cdot\vec{B}$, we have $\vec{A}=\vec{B}$. (5pts)

(b) Prove $\int_V \nabla\times\vec{F}dv=\oint_S d\vec{s}\times\vec{F}$, where the surface S encloses the volume V. Hint: Apply divergence theorem to $\nabla\cdot(\vec{F}\times\vec{c})$.

3. (15%) Let L be the operator on P_3, the set of all polynomials with degree less than 3, defined by

$$L(p(x)):=xp'(x)+p''(x).$$

(a) (6%) Find the matrix A representing L with respect to the ordered basis $[1,\ x,\ x^2]$.

(b) (9%) Let $p(x)=a_0+a_1x+a_2(1+x^2)$, where a_0, a_1, and a_2 are arbitrary real numbers. Please find the polynomial $L^n(p(x))$ in terms of a_0, a_1, a_2, and n.

4. (15%) Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{n\times r}$, and $C=AB$.

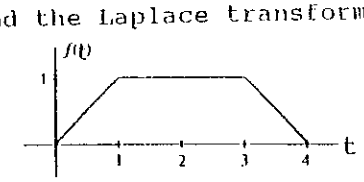
(a) (6%) Show that if A and B both have linearly independent column vectors, then the column vectors of matrix C will also be linearly independent.

(b) (9%) Is the converse statement "**if all columns of matrix C are linearly independent, then A and B both have linearly independent column vectors**" also true? Prove it if your answer is YES. Otherwise, give a counter example to show that it is WRONG.

5.(15%)Find the general solution of the following differential equation :

$$e^x\,dx+(e^x\cot y+2y\csc y)\,dy=0.$$

6.(10%) (a) Find the Laplace transform of the following function :



(10%)(b) Find the inverse Laplace transform of the following function :

$$\frac{s^2-4}{(s^2+4)^2}$$

科目：工程數學(乙)（電機工程學系碩士班 乙組）

1. (10%) A periodic function with a period 2 is defined by $x(t)=|t-1|$, $0\le t\le 2$. Expand $x(t)$ in Fourier series

$$x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}=a_0+\sum_{k=1}^{\infty}[a_k\cos(k\omega_0 t)+b_k\sin(k\omega_0 t)],$$ where ω_0 is the fundamental angular frequency. Determine

(a) $X[k]$ (5pts),

(b) $\sum_{k=1}^{\infty}a_k\cos(\frac{1}{2}k\omega_0)$. (5pts)

2. (15%) In the following $\vec{c}$ is an arbitrary constant vector.

(a) Show that if $\vec{c}\cdot\vec{A}=\vec{c}\cdot\vec{B}$, we have $\vec{A}=\vec{B}$. (5pts)

(b) Prove $\int_V \nabla\times\vec{F}dv=\oint_S d\vec{s}\times\vec{F}$, where the surface S encloses the volume V. Hint:

Apply divergence theorem to $\nabla\cdot(\vec{F}\times\vec{c})$.

3. (20%) Let L be the operator on P_2, the set of all polynomials with degree less than 2, defined by

$$L(p(x)) := 1+x+(x+5)p'(x).$$

(a) (6%) Find the matrix A representing L with respect to the ordered basis $E := [1,\ x]$.

(b) (14%) If $p(x)=c_0+c_1x$, where c_0 and c_1 are any real numbers, then $L^n(p(x))$ lies also in P_2. Let's denote $L^n(p(x)) := \alpha+\beta x$. Please find α and β in terms of c_0, c_1, and n.

4. (20%) Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{n\times r}$, and $C=AB$.

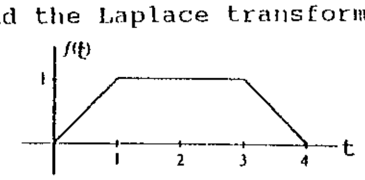
(a) (12%) Is the statement "**if all the columns of matrix C are linearly independent, then A and B both have linearly independent column vectors**" true? Prove it if your answer is YES. Otherwise, give explanations, based on the knowledge of linear algebra instead of explaining by a counter example, of why you think it is WRONG.

(b) (8%) Is the statement "**if A and B both have linearly independent column vectors, then all the columns of matrix C are linearly independent**" true? Prove it if your answer is YES. Otherwise, give explanations, based on the knowledge of linear algebra instead of explaining by a counter example, of why you think it is WRONG.

5.(15%)Find the general solution of the following differential equation:

$$e^x\,dx + (e^x\cot y + 2y\csc y)\,dy = 0.$$

6.(10%)(a) Find the Laplace transform of the following function :



(10%)(b) Find the inverse Laplace transform of the following function :

$$\frac{s^2-4}{(s^2+4)^2}$$

國立中山大學九十三學年度碩士班招生考試試題

科目：工數(甲)(電機所甲、丙選、丁、戊組) 共 1 頁 第 1 頁

1. (35%) In this problem, fill in the underlined blanks. Write you answers in the answer sheet. The detailed derivation is **NOT** required.

   Part I (15pts). Suppose $x(t)=t^2$, $-\pi<t<\pi$, and $x(t+2\pi)=x(t)$. Expand $x(t)$ in terms of Fourier series $x(t)=a_0+\sum_{k=1}^{\infty}[a_k\cos(k\omega_0 t)+b_k\sin(k\omega_0 t)]$, where ω_0 is the fundamental angular frequency.

   (a) What is a_k, $k=1,2,\cdots$? Answer ___(1)___.

   (b) What is b_k, $k=1,2,\cdots$? Answer ___(2)___.

   (c) Evaluate the series at $t=\pi$ to get $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=$ ___(3)___.

   Part II (10pts). The Fourier transform of $x(t)$ is defined as

   $$X(j\omega)\triangleq\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt. \quad \text{Suppose } x(t)=\begin{cases}1, & 0<t<\frac{1}{2}\\ 0, & t>\frac{1}{2}\end{cases} \text{ and } x(t) \text{ is odd.}$$

   (a) What is $X(j\omega)$? ___(4)___.

   (b) Compute $\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega=$ ___(5)___.

   Part III (10pts). Let field $\vec{A}=\vec{a}_r r^5$ exist in a ball of radius 1, where $\vec{r}=\vec{a}_x x+\vec{a}_y y+\vec{a}_z z$ is the position vector, $r=|\vec{r}|$, and $\vec{a}_r$ is the unit vector in $\vec{r}$ direction.

   (a) What is the volume integral $\int_V\nabla\cdot\vec{A}dv$ over the ball of unit radius? ___(6)___.

   (b) What is the surface integral $\int\vec{A}\cdot d\vec{s}$ over the surface of the top-half hemisphere of the ball of unit radius? ___(7)___.

2. (15%) Let $L: V\to W$ be a one-to-one and onto linear transformation between vector spaces V and W with $\dim(V)=r=\dim(W)$.
   Show that: if the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ is a basis for V, then the set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ is a basis for W.

3. (15%) Given $A=\begin{bmatrix}1 & 0 & 2 & -2\\ 0 & 1 & -1 & 3\\ -1 & 1 & -3 & 5\end{bmatrix}$ and let its column space be denoted by $R(A)$.

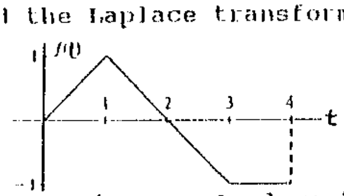
   (a) (6%) What is $R(A)^{\perp}$, the orthogonal complement of $R(A)$?

   (b) (9%) What is the projection matrix that will orthogonally project any vector $\mathbf{b}\in\mathbb{R}^3$ onto $R(A)^{\perp}$?

4. (15%) Find the general solution of the following differential equation :

   $$(-xy\sin x+2y\cos x)dx+(2x\cos x)dy=0$$

5. (10%) (a) Find the Laplace transform of the following function :



(10%) (b) Find the inverse Laplace transform of the following function :

$$\frac{s+12}{s^2+10s+35}$$

國立中山大學九十三學年度碩士班招生考試試題

科目：工數(乙)　(電機所乙組)　　共 1 頁 第 1 頁

1. (25%) In this problem, fill in the underlined blanks. Write you answers in the answer sheet. The detailed derivation is **NOT** required.

Part I (15pts). Suppose $x(t)=t^2$, $-\pi<t<\pi$, and $x(t+2\pi)=x(t)$. Expand $x(t)$ in terms of Fourier series $x(t)=a_0+\sum_{k=1}^{\infty}[a_k\cos(k\omega_0 t)+b_k\sin(k\omega_0 t)]$, where ω_0 is the fundamental angular frequency.

(a) What is a_k, $k=1,2,\cdots$? Answer __(1)__.

(b) What is b_k, $k=1,2,\cdots$? Answer __(2)__.

(c) Evaluate the series at $t=\pi$ to get $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=$ __(3)__.

Part II (10pts). The Fourier transform of $x(t)$ is defined as

$$X(j\omega)\triangleq\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt. \quad \text{Suppose } x(t)=\begin{cases}1, & 0<t<\frac{1}{2}\\ 0, & t>\frac{1}{2}\end{cases} \text{ and } x(t) \text{ is odd.}$$

(a) What is $X(j\omega)$? __(4)__.

(b) Compute $\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega=$ __(5)__.

2. (25%) Let V and W be vector spaces with $\dim(V)=r=\dim(W)$ and let L be a *linear* mapping from V to W, i.e. $L: V\to W$ is *linear*. Let $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ be a set of vectors in V with the corresponding image set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ lying in W. In each of the following questions, please first answer the question then prove the statement based on your answer.

(a) Under what condition (or conditions) of L will the statement "if the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ is linearly independent, then the set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ is linearly independent" hold?

(b) Under what condition (or conditions) of L will the statement "if the set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ is linearly independent, then the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ is linearly independent" hold?

(c) Suppose the condition (or conditions) proposed in (a) hold. Under what extra condition (or conditions) of L will the statement "if the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ is a basis for V, then the set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ is a basis for W" hold?

(d) Suppose the condition (or conditions) proposed in (b) hold. Under what extra condition (or conditions) of L will the statement "if the set $\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ is a basis for W, then the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ is a basis for V" hold?

Now, let's denote $E:=\{\mathbf{x}_1,\cdots,\mathbf{x}_r\}$ and $F:=\{\mathbf{y}_1,\cdots,\mathbf{y}_r\}$ as two bases for V and denote $G:=\{L(\mathbf{x}_1),\cdots,L(\mathbf{x}_r)\}$ and $H:=\{L(\mathbf{y}_1),\cdots,L(\mathbf{y}_r)\}$ as two bases for W. Let P be the transition matrix from basis E to basis F and let Q be the transition matrix from basis G to basis H. Denote A as the matrix representation of L with respect to bases E and G, and denote the matrix representation of L with respect to bases F and H by B. Please answer the next question without giving any proof.

(e) What are A and B, respectively, and what is the relationship between P and Q?

3. (15%) Let $A\in\mathbb{C}^{n\times n}$. Show that:

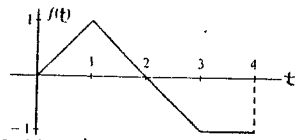
(a) (7%) if $\|A\mathbf{x}\|_2=\|\mathbf{x}\|_2$ for all $\mathbf{x}\in\mathbb{C}^n$ then $\langle A\mathbf{x},A\mathbf{y}\rangle=\langle\mathbf{x},\mathbf{y}\rangle$ for all $\mathbf{x},\mathbf{y}\in\mathbb{C}^n$.

(b) (8%) A is a unitary matrix if and only if $\|Ax\|_2=\|x\|_2$.

4.(15%)Find the general solution of the following differential equation :

$$(-xy\ \mathrm{Sin}\ x+2y\ \mathrm{Cos}\ x)dx+(2x\ \mathrm{Cos}\ x)dy=0$$

5.(10%) (a) Find the Laplace transform of the following function :



(10%)(b) Find the inverse Laplace transform of the function : $\dfrac{s+12}{s^2+10s+35}$

1. (25%)本題之計分僅以最後答案為準，不考慮計算過程。答案請寫在計算題部份，註明小題號，由(1)、(2)、…、依序列出。

Part I (15pts). A periodic signal $x(t)$ has a period $T=2$ and

$$x(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & \frac{1}{2}\le|t|<1\end{cases}.$$ Represent $x(t)$ in complex Fourier series $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{+jk\omega_0 t}$,

where ω_0 is the fundamental angular frequency. Find

(a) $X[0]=$ __(1)__. (b) $X[0]-X[1]+X[2]-\cdots=$ __(2)__. (c) $\sum_{k=-\infty}^{\infty}X^2[k]=$ __(3)__.

Part II (10pts). The Fourier transform of $x(t)$ is defined as

$$X(j\omega)\triangleq\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt.$$ Suppose $$x(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}.$$

(a) Find $X(j\pi)=$ __(4)__. (b) Calculate the following convolution integral

$$\int_{-\infty}^{\infty}\frac{\sin(\pi\tau)\sin[2\pi(t-\tau)]}{\pi^2\tau(t-\tau)}d\tau=\text{\_\_(5)\_\_}.$$

2. (40%) This problem contains two parts with 20 points in each part. ONLY ANSWERS WITHOUT PROOF are required in both parts.

Part I 是選擇並改正題。此部份中含有八個敘述，其中至少有四個敘述是錯誤的。請任選四個錯誤的敘述、並很精確地指出錯誤處、以及其正確的樣式。舉例如下：在下面三個敘述中

(S1) If $a>b$ and $b>c$, then $a\ge c$.

(S2) Let $A\in\mathbb{R}^{m\times n}$. Then matrix A is singular if and only if $\det(A)=0$.

(S3) Let $A\in\mathbb{R}^{m\times n}$. Then all columns of matrix A are linearly dependent if and only if null space $N(A)=\{\mathbf{0}\}$.

根據對數學和線性代數的了解，你知道(S1)是對的敘述、而(S2)和(S3)是錯誤的敘述。因此你寫的答案如下：

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answer for Part I:

wrong statement	correction (may be written in Chinese)
(S2)	The statement is true only when A is a square matrix, i.e. when $m=n$.
(S3)	$N(A)=\{\mathbf{0}\}$ should be corrected as $N(A)\ne\{\mathbf{0}\}$.

======

※※※注意：

1. 更正的敘述必須簡潔、精確；因為每一個敘述中錯誤處的更正都僅需書寫一行即可完成。
2. 當然同學注意到(S3)也可以更正成：將"only if part"敘述中的 linearly dependent 改成 linearly independent。因此為便於閱卷起見，統一規定：**當你認為某一個若且惟若的敘述是錯誤的，而且更改"only if part"或者"if part"中的敘述後即變為正確時，請更正"if part"中的敘述（意即：假設"only if part"中的敘述永遠是對的）**。不遵守此一規定的答案一律視為錯誤的答案。

此部份的分數計算方式為：

- 每一個正確的選擇為 2 分，如果你所提供的更正也正確、則可再得 3 分（因此，每一個正確的選擇及正確的更正共可得到 5 分，而如此答對四個正確的選擇及更正、便可得到 20 分）。如果選擇正確，但是所提供的更正不正確，則會因為錯誤的更正被扣 1 分，而僅得到 1 分。如果只寫了正確的選擇、但沒有提供任何更正、則得分數為 2。
- **每一個錯誤的選擇**(無論有否提供更正)**均會被扣 4 分**。然而，扣分的計算僅限於此部份的 20 分內，不會影響到 Part II 的得分數。

因此，如果你的答案如上所述，則可得到 10 分。但是，如果你的答案如下

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answer for Part I:

wrong statement	correction (may be written in Chinese)
(S1)	$a\ge c$ should be corrected as $a>c$.
(S3)	$N(A)=\{\mathbf{0}\}$ should be corrected as $N(A)\ne\{\mathbf{0}\}$.

則你僅會得到 1 分。其餘情況可自行類推。

科目：工程數學乙【電機系碩士班乙組】

Part I (20%): Let m and n be any two positive integers and let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. We use $R(A)$ to denote the range or column space of A and $N(A)$ to denote its null space. In the following (S1) to (S8) eight statements, at least four of them are wrong. Please choose arbitrary four statements that you think are WRONG and write the correction as simple and precise as possible in your answer.

(S1) Either exist $\mathbf{x} \in \mathbb{R}^n$ to satisfy the linear system $A\mathbf{x} = \mathbf{b}$ or exist $\mathbf{y} \in \mathbb{R}^m \cap N(A^T)$ such that $\langle \mathbf{y}, \mathbf{b} \rangle \neq 0$.

(S2) Either $\langle \mathbf{b}, A\mathbf{x} \rangle = 0$ for $\forall \mathbf{x} \in \mathbb{R}^n$ or $\langle \mathbf{b}, A\mathbf{x} \rangle > 0$ for every nonzero $\mathbf{x} \in \mathbb{R}^n$.

(S3) Matrix A has a right inverse, i.e. $\exists B \in \mathbb{R}^{n\times m}$ such that $AB = I_m$, if and only if A is full row rank, i.e. all rows of A are linearly independent. In that case, the linear system $A\mathbf{x} = \mathbf{b}$ is always consistent, i.e. it is always solvable.

(S4) The linear system $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent if and only if matrix A is full row rank.

(S5) Matrix $A^T A$ is nonsingular if and only if matrix A is full row rank. In that case, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution whenever it is consistent. When the linear system is inconsistent, however, it has a least squares solution described by the vector $\hat{\mathbf{x}} := (A^T A)^{-1} A^T \mathbf{b}$, i.e. $\| A\hat{\mathbf{x}} - \mathbf{b} \|_2 \le \| A\mathbf{x} - \mathbf{b} \|_2$ for $\forall \mathbf{x} \in \mathbb{R}^n$.

(S6) Suppose A is not a full rank matrix, i.e. assume $rank(A) = k < \min(m,n)$. Then $\exists Q \in \mathbb{R}^{m\times k}$ and $R \in \mathbb{R}^{k\times n}$ such that $A = QR$, where all columns of Q are orthonormal and R is a *non-square* upper triangular matrix.

(S7) From the QR factorization of a not full rank matrix A, i.e. $A = QR$ as mentioned in (S6), obviously we know $R(A) \subset R(Q)$. But the reverse inclusion $R(Q) \subset R(A)$ does not hold because of the existence of matrix R.

(S8) Let L be the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by A, i.e. $L(\mathbf{x}) := A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. Let L^* be the linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by A^T, i.e. $L^*(\mathbf{y}) := A^T\mathbf{y}$ for any $\mathbf{y} \in \mathbb{R}^m$. Then $\exists \mathbf{v} \in \mathbb{R}^n$ such that $L(\mathbf{v}) = \mathbf{b}$ if and only if $\mathbf{b} \in Ker(L^*)$.

Part II 含有四個子題，前三個子題是填充題，而第四個子題是選擇題。選擇題若有多於一個選項的答案，則全選對才給分。此部份答錯不倒扣。

Part II (20%): Let A be an $n \times n$ real matrix. Denote S and T as the symmetric and skew-symmetric parts of A, and let the eigenvalues of AA^T, S, and T be α_i, β_i, and γ_i for $i = 1, \cdots, n$, respectively. Please answer following questions without giving any proof.

(a) (6%) What are S and T respectively?

(b) (2%) What is the value of $trace(ST - TS)$?

(c) (6%) What is the relationship between $\sum_{i=1}^{n} \alpha_i$, $\sum_{i=1}^{n} \beta_i^2$, and $\sum_{i=1}^{n} \gamma_i^2$?

(d) (6%) Without loss of generality, suppose $n = 3$ and let $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$ be two pairs of distinct eigenvalues of S and T, respectively. Let $\mathbf{u}_i$ be a nonzero vector in the null space $N(S - \beta_i I)$ and, similarly, let $\mathbf{v}_i$ be a nonzero vector in the null space $N(T - \gamma_i I)$ for $i = 1, 2$, respectively. Then what are possible pairs of $\mathbf{u}_i$'s and $\mathbf{v}_i$'s?

(A) $\mathbf{u}_1 = [1\ 0\ 0]^T$, $\mathbf{u}_2 = [0\ 1\ -1]^T$ and $\mathbf{v}_1 = [1\ i\ -1]^T$, $\mathbf{v}_2 = [0\ 1\ -i]^T$

(B) $\mathbf{u}_1 = [1\ 0\ 0]^T$, $\mathbf{u}_2 = [0\ i\ 1]^T$ and $\mathbf{v}_1 = [i\ 1\ -1]^T$, $\mathbf{v}_2 = [0\ 1\ 1]^T$

(C) $\mathbf{u}_1 = [1\ 0\ -1]^T$, $\mathbf{u}_2 = [\sqrt{2}\ 1\ \sqrt{2}]^T$ and $\mathbf{v}_1 = [i\ 1\ 0]^T$, $\mathbf{v}_2 = [0\ 0\ 1]^T$

(D) $\mathbf{u}_1 = [1\ 0\ 1]^T$, $\mathbf{u}_2 = [-\frac{1}{2}\ 1\ \frac{1}{2}]^T$ and $\mathbf{v}_1 = [1\ 1\ 1]^T$, $\mathbf{v}_2 = [0\ -1\ 1]^T$

(E) All of above statements are correct.

(F) None of above statements is correct.

3. (15%) Find the general solution of $(x^2 + xy + y^2)\,dx - x^2\,dy = 0$.

4. (20%) Find the Laplace transform of the following function: $t\,e^{-t}\sinh 2t$

國立中山大學九十四學年度碩士班招生考試試題

科目：工程數學甲【電機系碩士班甲、丁、戊組(含丙組選考)】

1. (35%)本題之計分僅以最後答案為準，不考慮計算過程。答案請寫在計算題部份，註明小題號，由(1)、(2)、...、依序列出。

Part I (15pts). A periodic signal $x(t)$ has a period $T=2$ and

$x(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & \frac{1}{2}\le|t|<1\end{cases}$. Represent $x(t)$ in complex Fourier series $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{+jk\omega_0 t}$,

where ω_0 is the fundamental angular frequency. Find

(a) $X[0]=$ __(1)__. (b) $X[0]-X[1]+X[2]-\cdots=$ __(2)__. (c) $\sum_{k=-\infty}^{\infty}X^2[k]=$ __(3)__.

Part II (10pts). The Fourier transform of $x(t)$ is defined as

$X(j\omega)\triangleq\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$. Suppose $x(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$.

(a) Find $X(j\pi)=$ __(4)__. (b) Calculate the following convolution integral

$\int_{-\infty}^{\infty}\frac{\sin(\pi\tau)\sin[2\pi(t-\tau)]}{\pi^2\tau(t-\tau)}d\tau=$ __(5)__.

Part III (10pts). Let field $\vec{G}(x,y,z)=\vec{a}_x(x-yz)+\vec{a}_y(y^2-zx)+\vec{a}_z(z^2-xy)$.

(a) Find the line integral $\int_{C_1}\vec{G}(x,y,z)\cdot d\vec{l}=$ __(6)__, where C_1 is a segment of the curve $y=x^2$, $z=x$ from (0,0,0) to (1,1,1).

(b) Compute the sum of line integrals $\int_{C_2}\vec{G}(x,y,z)\cdot d\vec{l}+\int_{C_3}\vec{G}(x,y,z)\cdot d\vec{l}$ where C_2 is the straight line from (1,1,1) to (0,0,1) and C_3 is along the z-axis from (0,0,1) to (0,0,0).

Answer: __(7)__.

2. (30%) This problem contains two parts with 15 points in each part. ONLY ANSWERS WITHOUT PROOF are required in both parts.

Part I 是選擇並改正題。此部份中含有六個敘述，其中至少有三個敘述是錯誤的。請任選三個錯誤的敘述、並很精確地指出錯誤處、以及其正確的樣式。舉例如下：在下面三個敘述中

(S1) If $a>b$ and $b>c$, then $a\ge c$.

(S2) Let $A\in\mathbb{R}^{m\times n}$. Then matrix A is singular if and only if $\det(A)=0$.

(S3) Let $A\in\mathbb{R}^{m\times n}$. Then all columns of matrix A are linearly dependent if and only if null space $N(A)=\{\mathbf{0}\}$.

根據對數學和線性代數的了解，你知道(S1)是對的敘述、而(S2)和(S3)是錯誤的敘述。因此你寫的答案如下：

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answer for Part I:

wrong statement	correction (may be written in Chinese)
(S2)	The statement is true only when A is a square matrix, i.e. when $m=n$.
(S3)	$N(A)=\{\mathbf{0}\}$ should be corrected as $N(A)\ne\{\mathbf{0}\}$.

======

※※※注意：

1. 更正的敘述必須簡潔、精確；因為每一個敘述中錯誤處的更正都僅需書寫一行即可完成。
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此部份的分數計算方式為：

- 每一個正確的選擇為 2 分，如果你所提供的更正也正確、則可再得 3 分 (因此，每一個正確的選擇及正確的更正共可得到 5 分，而如此答對四個正確的選擇及更正、便可得到 20 分)。如果選擇正確，但是所提供

的更正不正確，則會因為錯誤的更正被扣 1 分，而僅得到 1 分。如果只寫了正確的選擇、但沒有提供任何更正、則得分數為 2。

- 每一個錯誤的選擇(無論有否提供更正)均會被扣 4 分。然而，扣分的計算僅限於此部份的 20 分內，不會影響到 Part II 的得分數。

因此，如果你的答案如上所述，則可得到 10 分。但是，如果你的答案如下

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answer for Part I:

wrong statement correction (may be written in Chinese)

(S1) $a \ge c$ should be corrected as $a > c$.

(S3) $N(A) = \{0\}$ should be corrected as $N(A) \ne \{0\}$.

======

則你僅會得到 1 分。其餘情況可自行類推。

Part I (15%): Let m and n be any two positive integers and let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. We use $R(A)$ to denote the range or column space of A and $N(A)$ to denote its null space. In the following (S1) to (S6) six statements, at least three of them are wrong. Please choose arbitrary three statements that you think are WRONG and write the correction as simple and precise as possible in your answer.

(S1) Either exist $\mathbf{x} \in \mathbb{R}^n$ to satisfy the linear system $A\mathbf{x} = \mathbf{b}$ or exist $\mathbf{y} \in \mathbb{R}^m \cap N(A^T)$ such that $\langle \mathbf{y}, \mathbf{b} \rangle \ne 0$.

(S2) Either $\langle \mathbf{b}, A\mathbf{x} \rangle = 0$ for $\forall \mathbf{x} \in \mathbb{R}^n$ or $\langle \mathbf{b}, A\mathbf{x} \rangle > 0$ for each nonzero $\mathbf{x} \in \mathbb{R}^n$.

(S3) Matrix A has a right inverse, i.e. $\exists B \in \mathbb{R}^{n\times m}$ such that $AB = I_m$, if and only if A is full row rank, i.e. all rows of A are linearly independent. In that case, the linear system $A\mathbf{x} = \mathbf{b}$ is always consistent, i.e. it is always solvable.

(S4) The linear system $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent if and only if matrix A is full row rank.

(S5) Matrix $A^T A$ is nonsingular if and only if matrix A is full row rank. In that case, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution whenever it is consistent. When the linear system is inconsistent, however, it has a least squares solution described by the vector $\hat{\mathbf{x}} := (A^T A)^{-1} A^T \mathbf{b}$, i.e. $\| A\hat{\mathbf{x}} - \mathbf{b} \|_2 \le \| A\mathbf{x} - \mathbf{b} \|_2$ for $\forall \mathbf{x} \in \mathbb{R}^n$.

(S6) Let L be the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by A, i.e. $L(\mathbf{x}) := A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. Let L^* be the linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by A^T, i.e. $L^*(\mathbf{y}) := A^T\mathbf{y}$ for any $\mathbf{y} \in \mathbb{R}^m$. Then $\exists \mathbf{v} \in \mathbb{R}^n$ such that $L(\mathbf{v}) = \mathbf{b}$ if and only if $\mathbf{b} \in Ker(L^*)$.

Part II 含有三個子題，它們都是填充題。此部份答錯不倒扣。

Part II (15%): Let A be an $n\times n$ real matrix. Denote S and T as the symmetric and skew-symmetric parts of A, and let the eigenvalues of AA^T, S, and T be α_i, β_i, and γ_i for $i = 1, \cdots, n$, respectively. Please answer following questions without giving any proof.

(a) (6%) What are S and T respectively?

(b) (2%) What is the value of $trace(ST - TS)$?

(c) (7%) What is the relationship between $\sum_{i=1}^{n} \alpha_i$, $\sum_{i=1}^{n} \beta_i^2$, and $\sum_{i=1}^{n} \gamma_i^2$?

3. (15%) Evaluate $\oint_C \dfrac{z+8}{z^2+z-2} dz$ by using the residue theorem,

where C： $|z| = 3$ and $z = x + iy$

4.(20%) Find the general solution of $(x^2 + xy + y^2)\,dx - x^2\,dy = 0$.

國立中山大學95學年度碩士班招生考試試題

1. (35%)填空題，計分僅以最後答案為準，不考慮計算過程。答案請寫在答案卷$\boxed{\text{計算題}}$部份，註明小題號，按(1)、(2)、…、依序列出。

Part I (25pts). The Fourier transform (FT) of $x(t)$ is defined as

$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$. Let $w(t)=1$ be a constant function. Define

$y(t)=\begin{cases}1, & t\geq 0\\ -1, & t<0\end{cases}$ and unit step function $u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$.

(a) Find the FTs: $W(j\omega)=$__(1)__, $Y(j\omega)=$__(2)__ and $U(j\omega)=$__(3)__.

(b) Express the FT $Z(j\omega)=$__(4)__ of $z(t)=\int_{-\infty}^{t} x(\tau)d\tau$ in terms of $X(j\omega)$.

(c) Evaluate the integral $\int_{-\infty}^{\infty} U(j\omega)d\omega=$__(5)__.

Part II (10pts). The spherical coordinates (r,θ,ϕ) is related to the rectangular coordinates by $r=\sqrt{x^2+y^2+z^2}$, $\theta=\tan^{-1}(\sqrt{x^2+y^2}/z)$, $\phi=\tan^{-1}(y/x)$. Let $\vec{F}=\vec{a}_r 1+\vec{a}_\theta 2-\vec{a}_\phi 3$.

(a) Find the surface integral $\int_S \vec{F}\cdot d\vec{s}=$__(6)__ where S is a portion of an upper sphere ($z>0$) with $\theta\geq\pi/3$. The radius of the sphere is 2.

(b) Find $F_x=$__(7)__, the x-component of $\vec{F}$ at a point whose spherical coordinates is $(2,\pi/4,\pi/4)$.

2. (30%) Let $V=\mathbb{R}^{n\times n}$, $W=\{\mathbf{w}\in V \mid \mathbf{w}=\mathbf{w}^T\}$, and let L be a transformation from V to W. Define a scalar-valued function $\langle\cdot,\cdot\rangle: V\times V\to\mathbb{R}$ by $\langle\mathbf{u},\mathbf{v}\rangle := tr(\mathbf{u}^T\mathbf{v})$.

(a) (4%) Show that L defined by, for any $\mathbf{v}\in V$, $L(\mathbf{v}) := \frac{1}{2}(\mathbf{v}+\mathbf{v}^T)$ is linear and onto.

(b) (4%) Show that $(V,\langle\cdot,\cdot\rangle)$ is an inner product space.

(c) (6%) Derive the orthogonal complement $W^\perp$.

Now let's consider $n=2$ case. Let E and F be the orthonormal bases, generated from the ordered basis $\hat{E}=[\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}]$ for V and the ordered basis $\hat{F}=[\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}a&b\\b&c\end{bmatrix}]$ for W, respectively. For simplicity, assume that all entries of vectors in E and F are nonnegative.

(d) (8%) Derive the matrix A to represent L with respect to the orthonormal bases E and F.

(e) (8%) Find the set of all vectors $\mathbf{v}\in V$, denoted by $\mathbf{v}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ with $a,b,c,d\in\mathbb{R}$, having the property that the angle between $\mathbf{v}$ and its image $\mathbf{w}$ in W is 45°, i.e. $\measuredangle(\mathbf{v},\mathbf{w})=\pi/4$.

3. (15%) Evaluate $\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$, by using the residue theorem,

where C : $|z|=4$.

4. (20%) Find the general solution of the differential equation: $x^2(y'-1)=y^2$

1. (25%)填空題，計分僅以最後答案為準，不考慮計算過程。答案請寫在答案卷$\boxed{\text{計算題}}$部份，註明小題號，按(1)、(2)、...、依序列出。

The Fourier transform (FT) of $x(t)$ is defined as

$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$. Let $w(t) = 1$ be a constant function. Define

$y(t) = \begin{cases} 1, & t \ge 0 \\ -1, & t < 0 \end{cases}$ and unit step function $u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$.

(a) Find the FTs: $W(j\omega)$=__(1)__, $Y(j\omega)$=__(2)__ and $U(j\omega)$=__(3)__.

(b) Express the FT $Z(j\omega)$=__(4)__ of $z(t) = \int_{-\infty}^{t} x(\tau)d\tau$ in terms of $X(j\omega)$.

2. (40%) Let $L: V \to W$ be a linear transformation between vector spaces V and W with $\dim V = p$ and $\dim W = q$, and let $E = [\mathbf{v}_1, \cdots, \mathbf{v}_p]$ and $F = [\mathbf{w}_1, \cdots, \mathbf{w}_q]$ be two ordered bases for V and W, respectively. Let A be the matrix representation of L with respect to bases E and F.

(a) (10%) Show that L is onto if and only if matrix A is full row rank.

Now let's restrict $V = \mathbb{R}^{n\times n}$ and $W = \{\mathbf{w} \in V \mid \mathbf{w} = \mathbf{w}^T\}$, and let L be a transformation from V to W. Define a scalar-valued function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ by $\langle \mathbf{u}, \mathbf{v}\rangle := tr(\mathbf{u}^T\mathbf{v})$.

(b) (4%) Show that L defined by, for any $\mathbf{v} \in V$, $L(\mathbf{v}) := \frac{1}{2}(\mathbf{v} + \mathbf{v}^T)$ is linear and onto.

(c) (4%) Show that $(V, \langle\cdot,\cdot\rangle)$ is an inner product space.

(d) (6%) Derive the orthogonal complement $W^{\perp}$.

Now let's consider $n = 2$ case. Let E and F be the orthonormal bases, generated from the ordered basis $\hat{E} = [\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}]$ for V and the ordered basis $\hat{F} = [\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}a&b\\b&c\end{bmatrix}]$ for W, respectively. For simplicity, assume that all entries of vectors in E and F are nonnegative.

(e) (8%) Derive the matrix A to represent L with respect to the orthonormal bases E and F.

(f) (8%) Find the set of all vectors $\mathbf{v} \in V$, denoted by $\mathbf{v} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$, having the property that the angle between $\mathbf{v}$ and its image $\mathbf{w}$ in W is 45°, i.e. $\measuredangle(\mathbf{v}, \mathbf{w}) = \pi/4$.

3. (15%) Find the general solution of the differential equation: $x^2(y' - 1) = y^2$

4. (20%) Find the Laplace transform of the function: $(1 - \cos t)/t$.

國立中山大學９６學年度碩士班招生考試試題

科目：工程數學甲【電機系碩士班甲、丁、戊、庚組；丙組選考】 共 1 頁 第 1 頁

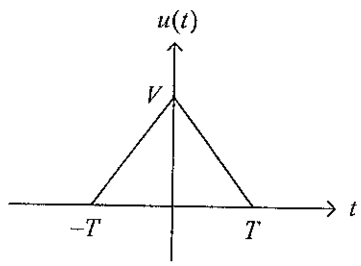
1. (15%) Find the general solution of the following differential equation

$$(y\cos x - \sin 2x)dx + dy = 0.$$

2. (20%) Evaluate $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3}$ by using the residue theorem.

3. (15%) Let S be the surface of the paraboloid $x^2+y^2+z = 2$ lying above the xy-plane. Then the area of S = ______. （只須寫出答案，不須列出算式，計分僅以答案為準。）

4. (20%) （此題只須寫出答案，不須列出算式，計分僅以答案為準。）Consider the signal $u(t)$ described by the following plot



(a) (7%) The Laplace transform of $u(t)$ is defined by $U(s) := \int_0^{\infty} u(t)e^{-st}dt$. So, $U(s)$ = ______.

(b) (6%) The Fourier transform of $u(t)$ is defined by $U(w) := \int_{-\infty}^{\infty} u(t)e^{-jwt}dt$. So, $U(w)$ = ______.
（答案必須化成最簡式，否則不予計分。）

(c) (7%) Suppose V and T be chosen such that $\int_{-\infty}^{\infty} U(w)dw = \int_{-\infty}^{\infty} |U(w)|^2 dw$. Then the relationship between V and T is ______.

5. (30%) Let $A = \begin{bmatrix} t & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 4 & -5 \\ 1 & 2 & -1 \end{bmatrix}$, where t is an undetermined real parameter.

(a) (6%) Please describe the null space $N(A)$ with respect to the value of parameter t.

From now on, based on the result of (a), let the value of t be chosen so that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solution.

(b) (8%) Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Let M be a nonsingular matrix so that $MA = A_{rref}$ and $M\mathbf{b} = [\alpha\ \beta\ \gamma\ \delta]^T \in \mathbb{R}^4$, where A_{rref} denotes the reduced row echelon form obtained from matrix A after the application of Gauss elimination. Please find the set of $[\alpha\ \beta\ \gamma\ \delta]^T$ such that equation $A\mathbf{x} = \mathbf{b}$ is solvable, and moreover, write out the solution set, described by S, of the equation.

(c) (6%) Please find $\mathbf{x}^*$ to solve the optimization problem $\min_{\mathbf{x}\in S} f(\mathbf{x})$ with $f(\mathbf{x}) := \|\mathbf{x}\|_2$.

(d) (10%) Let P denote the projection matrix to orthogonally project any vector from $\mathbb{R}^4$ into $R(A)^{\perp}$, the orthogonal complement of the range of A. Please find P and write out the set of all its eigenvalues.

國立中山大學９６學年度碩士班招生考試試題

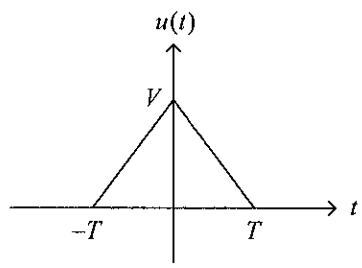
科目：工程數學乙【電機系碩士班乙組】

1. (15%) Find the general solution of the following differential equation

$$(x-y)dx+(x+y)dy = 0.$$

2. (20%) Evaluate $\int_{-\infty}^{\infty}\frac{x\,dx}{(x-1)(x^2+2x+2)}$ by using the residue theorem.

3. (20%)（此題只須寫出答案，不須列出算式，計分僅以答案為準。）Consider the signal $u(t)$ described by the following plot



(a) (7%) The Laplace transform of $u(t)$ is defined by $U(s) := \int_0^{\infty} u(t)e^{-st}dt$. So, $U(s) =$ _______.

(b) (6%) The Fourier transform of $u(t)$ is defined by $U(w) := \int_{-\infty}^{\infty} u(t)e^{-jwt}dt$. So, $U(w) =$ _______.
（答案必須化成最簡式，否則不予計分。）

(c) (7%) Suppose that V and T are chosen such that $\int_{-\infty}^{\infty} U(w)dw = \int_{-\infty}^{\infty} |U(w)|^2 dw$. Then the relationship between V and T is _______.

4. (30%) Let $A = \begin{bmatrix} t & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 4 & -5 \\ 1 & 2 & -1 \end{bmatrix}$, where t is an undetermined real parameter.

(a) (6%) Please describe the null space $N(A)$ with respect to the value of parameter t.

From now on, based on the result of (a), let the value of t be chosen so that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solution.

(b) (8%) Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Let M be a nonsingular matrix so that $MA = A_{rref}$ and $M\mathbf{b} = [\alpha\ \beta\ \gamma\ \delta]^T \in \mathbb{R}^4$, where A_{rref} denotes the reduced row echelon form obtained from matrix A after the application of Gauss elimination. Please find the set of $[\alpha\ \beta\ \gamma\ \delta]^T$ such that equation $A\mathbf{x} = \mathbf{b}$ is solvable, and moreover, write out the solution set, described by S, of the equation.

(c) (6%) Please find $\mathbf{x}^*$ to solve the optimization problem $\min_{\mathbf{x}\in S} f(\mathbf{x})$ with $f(\mathbf{x}) := \|\mathbf{x}\|_2$.

(d) (10%) Let P denote the projection matrix to orthogonally project any vector from $\mathbb{R}^4$ into $R(A)^{\perp}$, the orthogonal complement of the range of A. Please find P and write out the set of all its eigenvalues.

5. (15%) Let $A \in \mathbb{R}^{n\times n}$ be such that the product operation between A and its transpose is commutative. Please answer the next two questions followed by detailed proofs.

(a) (6%) What is the relationship between $N(A)$ and $N(A^T)$?

(b) (9%) Suppose that α and β are two distinct real eigenvalues of A with $\mathbf{x}$ and $\mathbf{y}$ being the corresponding eigenvectors, respectively. What kind of relationship between $\mathbf{x}$ and $\mathbf{y}$ can be derived from the result of (a) ?

1. (10%) Solve the initial value problem (IVP) for the following ODE

$$y'' + y = 5x + 8\sin x, \quad y(\pi) = 0,\ y'(\pi) = 2.$$

2. (15%) Find the solution of the initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

3. (5%) Find the Laplace transform of (a) $\mathcal{L}\{e^{-5t}\}$ (b) $\mathcal{L}\{\sin 3t\}$

4. (15%) Solve the initial value problem by the Laplace transform

$$\begin{cases} y_1' + 2y_2' = 1 \\ 3y_1' + y_2' + y_2 = t \end{cases} \quad y_1(0) = 0,\ y_2(0) = 0$$

5. (15%) Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$ in a Fourier series.

6. (13%) Evaluate the following integral

$$\oint_C \frac{dz}{\sinh\ (2z)},$$

where z is a complex variable and C denotes the circle $|z| = 2$ described in the positive sense.

7. (15%) The set

$$S = \left\{ \frac{1}{\sqrt{2}},\ \cos\ x,\ \cos\ 2x,\ \cos\ 3x,\ \cos\ 4x \right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined as

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx,$$

where $C[-\pi, \pi]$ is the set of all functions f that are continuous on $[-\pi, \pi]$. Suppose that the function $\sin^4 x$ can be written in a linear combination of elements of S as

$$\sin^4 x = \frac{3\sqrt{2}}{8}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}(\cos\ 2x) + \frac{1}{8}(\cos\ 4x).$$

Use the above equation and orthogonal basis property (but do not compute antiderivatives, otherwise you will get zero credit), find the values of the following integrals:

(i) $\int_{-\pi}^{\pi} \sin^4 x\, dx$ (ii) $\int_{-\pi}^{\pi} \sin^4 x \cos(3x)\, dx$ (iii) $\int_{-\pi}^{\pi} \sin^4 x \cos(4x)\, dx$

8. (12%) Let P_4 be the set of all polynomials of degree less than 4. In P_4 the inner product is defined by

$$\langle p, q \rangle = \sum_{i=1}^{4} p(x_i) q(x_i),$$

where $x_i = (i-2)/2$ for $i = 1, \cdots, 4$. Its norm is defined by

$$\| p \| = \sqrt{\langle p, q \rangle} = \left\{ \sum_{i=1}^{4} [p(x_i)]^2 \right\}^{1/2}.$$

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Compute (a) $\| x^2 \|$, (b) the distance between x and x^2.

1. (15%) Find the solution of the initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

2. (5%) Find the Laplace transform of (a) $\mathcal{L}\{e^{-5t}\}$ (b) $\mathcal{L}\{\sin 3t\}$

3. (15%) Solve the initial value problem by the Laplace transform

$$\begin{cases} y_1' + 2y_2' = 1 \\ 3y_1' + y_2' + y_2 = t \end{cases} \quad y_1(0) = 0, \; y_2(0) = 0$$

4. (15%) Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$ in a Fourier series.

5. (10%) Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

6. (13%) Find the best quadratic polynomial to fit the data $p(-1) = 0$, $p(0) = 1$, $p(1) = 2$, $p(2) = 4$.

7. (15%) The set

$$S = \left\{ \frac{1}{\sqrt{2}}, \; \cos x, \; \cos 2x, \; \cos 3x, \; \cos 4x \right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined as

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx,$$

where $C[-\pi, \pi]$ is the set of all functions f that are continuous on $[-\pi, \pi]$. Suppose that the function $\sin^4 x$ can be written in a linear combination of elements of S as

$$\sin^4 x = \frac{3\sqrt{2}}{8} \left(\frac{1}{\sqrt{2}} \right) - \frac{1}{2} (\cos 2x) + \frac{1}{8} (\cos 4x).$$

Use the above equation and orthogonal basis property (but do not compute antiderivatives, otherwise you will get zero credit), find the values of the following integrals:

(i) $\int_{-\pi}^{\pi} \sin^4 x \, dx$ (ii) $\int_{-\pi}^{\pi} \sin^4 x \cos(3x) \, dx$ (iii) $\int_{-\pi}^{\pi} \sin^4 x \cos(4x) \, dx$

8. (12%) Let P_4 be the set of all polynomials of degree less than 4. In P_4 the inner product is defined by

$$\langle p, q \rangle = \sum_{i=1}^{4} p(x_i) q(x_i),$$

where $x_i = (i-2)/2$ for $i = 1, \cdots, 4$. Its norm is defined by

$$\| p \| = \sqrt{\langle p, q \rangle} = \left\{ \sum_{i=1}^{4} [p(x_i)]^2 \right\}^{1/2}.$$

387

Compute (a) $\| x^2 \|$, (b) the distance between x and x^2.