中央大學

土木工程學系

91~97 學年度
工程數學考古題

一、

波方程 $u_{xx}+u_{yy}=u_{tt}$ 的解滿足以下條件：

B.C.： $u(0,y,t)=0$ ， $u(1,y,t)=0$ ， $u(x,0,t)=0$ ， $u(x,1,t)=0$

I.C.： $u(x,y,0)=0$ ， $u_t(x,y,0)=g(x,y)$

設 $g(x,y)$ 爲已知函數而此解可以表示成

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}D_{mn}\sin(m\pi x)\sin(n\pi y)\sin(K_{mn}t) \text{ .}$$

請找出 D_{mn} 和 K_{mn} 的數學表達式。(20%)

二、

若 C 代表複平面上的圓 $|Z|=4$ ，請計算 $\oint_C \dfrac{dZ}{e^Z-e^{-Z}}=?$ (20%)

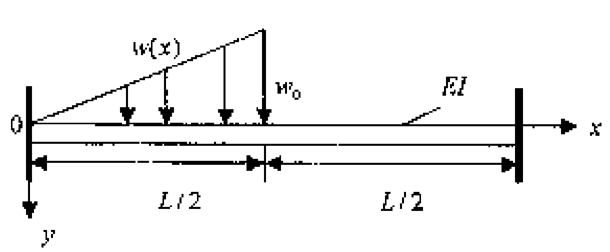
三、

如圖所示之彈性梁，承受一不連續分布之荷載 $w(x)$ ，根據材料力學理論，此一梁之撓曲線 $y(x)$ 之控制方程式爲 $EI\dfrac{d^4y}{dx^4}=w(x)$

(甲) 請用單位階梯函數(unit step function)， $u(t-a)=0$ ， $0\le t<a$ ；
$u(t-a)=1$ ， $t\ge a$ ，表示此一不連續分布之荷載函數 $w(x)$? (5%)

(乙) 此一梁撓曲問題求解之邊界條件是什麼? (5%)

(丙) 請用 Laplace 轉換法，求解此一梁在 $w(x)$ 荷載作用下之撓曲線 $y(x)=$? (10%)



注意：背面有試題

四、

（甲）已知函數 $F(x,y,z)=axy^2+byz+cz^2x^3$ 在點(1,2,-1)處沿著 Z 軸的方向有最大的方向導數(directional derivative)，其值為 64，請問 a、b、c 三個常數值分別為何？（9%）

（乙）請利用 $u=2xy$，$v=x^2-y^2$ 的變數變換，計算下面的積分值。(15%)

$$\iint_R (x^2+y^2)(x^2-y^2)^{1/3}\,dA$$ 其中 R 是由 x=0，x=1，y=0，y=1 所圍成的區域。

五、

（甲）矩陣 A 的伴隨矩陣(adjoint matrix)表示為 adj(A)，其行列式值表為｜adj(A)｜，請列式計算（｜adj(A)｜)^3，其中 A 為 $\begin{bmatrix} 5 & 7 & 3 & 4 & 1 \\ 4 & 8 & 3 & 4 & 1 \\ 4 & 7 & 4 & 4 & 1 \\ 4 & 7 & 3 & 5 & 1 \\ 4 & 7 & 3 & 4 & 2 \end{bmatrix}$ (8%)

（乙）已知矩陣[A]的特徵值為 1 和 6，所對應的特徵向量分別為 $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ 和 $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$，

其反矩陣的平方可寫成 $\left([A]^{-1}\right)^2=\dfrac{1}{Q}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$，請問 a、b、c、d 和 Q 之值為何？

(8%)

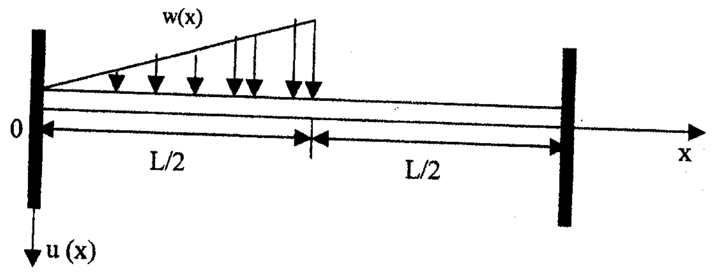
系所別：土木工程學系　甲組丙組科目：工程數學

1.

如下圖所示之彎曲剛度爲 EI 之梁，在 0 <x<L/2，承受一荷載 $w(x)=\dfrac{2w_0 x}{L}$ ，當 L> x>L/2 時 w(x)=0。此一梁撓曲變形 u(x)之控制方程爲

$EI\dfrac{d^4u}{dx^4}=w(x)$,　　邊界條件爲 $u(0)=\dfrac{du(o)}{dx}=u(L)=\dfrac{du(L)}{dx}=0$

請用 Laplace 轉換求解此一梁之撓曲變形 u(x)=?　(25%)



2.

請求解以下聯立微分方程

$$2\frac{dy_1}{dt}-\frac{dy_2}{dt}-\frac{dy_3}{dt}=0$$
$$\frac{dy_1}{dt}+\frac{dy_2}{dt}=4t+2 \qquad y_1(0)=y_2(0)=y_3(0)=0$$
$$\frac{dy_2}{dt}+y_3=t^2+2$$

(10 %)

3. 方陣[A]滿足 $-[A]^3+6[A]^2-11[A]+6[I]=[0]$ ，計算方陣[A]之反矩陣之特徵值的和(10%)

4. 已知流體的流速爲 $\vec{V}=(x^2+\cos y)\vec{i}+(x^3z^2)\vec{j}+(y^6)\vec{k}$ ，試問此流體每單位時間流經由 $y=x^2$ ，$z=9-y$ ，$z=0$ 所圍成的表面 S 的流量。(20%)

注意：背面有試題

系所別：土木工程學系 甲組丙組科目：工程數學

5. 考慮複變數函數 $f(z)=\dfrac{1}{(i+1-z)(z-i-2)}$，$z=x+iy$，$i=\sqrt{-1}$。以 $z=i$ 為中心作展開可得出 Laurent series 表示式 $f(z)=\sum_{n=-\infty}^{\infty} C_n (z-i)^n$。若此 series 的收斂區間為 $1<|z-i|<2$，請計算出 series 中的係數 C_2 和 C_{-2} 的數值。（20分）

6. 熱方程 $u_{xx}=u_t$ 的解 $u(x,t)$ 滿足初始條件 $u(x,0)=f(x)$。若此解在邊界 $x=0$ 和 $x=L$ 處滿足邊界條件 $u_x(0,t)=0$ 和 $u(L,t)-u_x(L,t)=0$，請求出 $u(x,t)$。

（15分）

參考用

國立中央大學九十三學年度碩士班研究生入學試題卷　共 1 頁　第 1 頁

所別：土木工程學系碩士班甲、丙組　科目：工程數學

1) 設 $\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$ 代表 $f(x)$ 的 Fourier Transform，若 $f(x)=\frac{x}{x^2+a^2}$ $(a>0)$，請算出 $\hat{f}(\omega)$。

(15分)

2) 請寫出偏微分方程 $\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}=0$ 的特徵線方程。並請指出此方程是屬於那一種類型(例如拋物型，等等)。請分 $x>0$ 和 $x<0$ 兩种情況討論。

(20分)

3) 函數 $f(t)$ 之 Laplace 轉換為 $F(s)=\ln(1+\frac{\omega^2}{s^2})$，請求解 $f(t)=?$

(15分)

4) 求解微分方程 $(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr})^2\phi=0$ 之通解 $\phi(r)=?$

(20分)

5) 將二次方程式 $5x^2+3y^2+3z^2-2xy+2yz-2xz=1$ 轉換成 $ax'^2+by'^2+cz'^2=d$ 的形式時，abc 的乘積？

(15分)

6) 有一曲線 $x=3\cos t$，$y=3\sin t$，$z=4t$，則這一曲線的曲率半徑為何？

(15分)

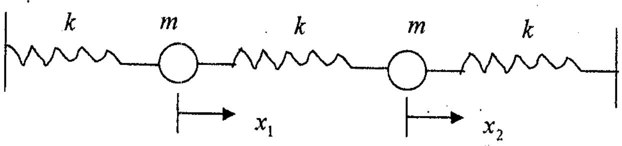
參考用

國立中央大學九十三學年度碩士班研究生入學試題卷 共 頁 第 頁

1. (a)(5%) 請問如圖所示之彈簧-質點系統之運動方程式是否為以下之型式？

$$m\ddot{x}_1 = -kx_1 - k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 + k(x_2 - x_1)$$

(b) (10%) 請求此一系統之特徵值及特徵向量？



2. (15%) 請求取以下積分

$$I = \int_S \vec{n} \bullet \vec{\nabla} \times \vec{v} ds$$

其中 $\vec{v} = x^3\vec{j} - (z+1)\vec{k}$ ，S 是曲面 $z = 4 - 4x^2 - y^2$ 之介於 z= 0 與 z= 4 之間的曲面，$\vec{n}$ 是曲面之外法向量，$\vec{j}$，$\vec{k}$ 為坐標軸 y 與 z 軸之單位向量。

3) (30%) 考慮在無窮長的鐵棒中的熱傳導問題，其控制方程為 $k u_{xx} = u_t$ (k為已知常數)。設 $u(x,t)$ 在定義域中為有限值，而其滿足的初始條件為 $u(x,0) = f(x)$，$-\infty < x < \infty$，請計算出 $u(x,t)$。

4) (20%) 請計算出以下積分：

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{a^2 + \omega^2} d\omega ,$$

其中 $i = \sqrt{-1}$。

注意：背面有試題

~~所別：土木工程學系碩士班~~ 科目：~~工程數學~~

所別：數學系碩士班 不分組 科目：數值分析

5) (20%) 如下列之微分方程式：

$$y'' + 2y' + 5y = f(t), \quad y(0)=0, \ y'(0)=1,$$

其中
$$f(t) = \begin{cases} t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

試求 $y(t) = ?$, $y(\frac{\pi}{2}) = ?$

所別：土木工程學系碩士班 甲組 丙組 科目：工程數學

1. (25%)

1) Show that { 1, x, x^2+2x } form a basis of $P_2[-1,1]$.

where $P_2[-1,1] = \{ f(x) \mid f(x) = a_0+a_1x+a_2x^2, \forall\, a_0,a_1,a_2 \in R,\ x\in[-1,1] \}$.

2) With an inner product defined by

$$(f,g) = \int_{-1}^{1} (x+1)\, f(x) g(x)\, dx$$

construct an orthonormal basis for $P_2[-1,1]$, starting with the above basis.

2. (30%)

長度為 L 的簡支樑其位移 $u(x,t)$ 如附圖所示。$u(x,t)$ 滿足偏微分方程

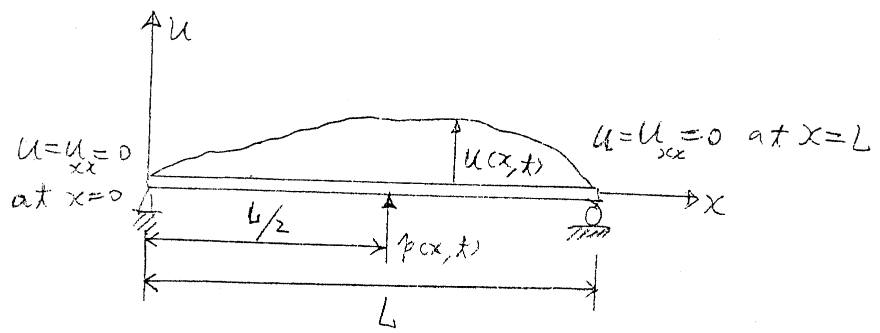
$$EI \frac{\partial^4 u}{\partial x^4} + m \frac{\partial^2 u}{\partial t^2} = p(x,t),$$

其中 E, I, m 皆為常數，而外力 $p(x,t)=\begin{cases} 0, & t\le 0 \\ 5\delta(x-\frac{L}{2}), & t>0 \end{cases}$

此處 δ 代表 Delta 函數。已知樑中央 $(x=\frac{L}{2})$ 處的位移為

$$u(\tfrac{L}{2},t) = \frac{L^3}{\pi^4 EI} \sum_{n=1,3,5,\ldots}^{\infty} g_n(t)$$

而且 $t\le 0$ 時 $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$，請問 $g_n(t) = ?$



注意：背面有試題

所別：土木工程學系碩士班 甲組 科目：工程數學
丙組

3.(25%)

用 Fourier 轉換或其他方法求以下積分：

$$\int_{-\infty}^{\infty} e^{i\omega x} d\omega = ? \qquad (-\infty < x < \infty)$$

上式中 $i = \sqrt{-1}$。

4. (20%) 若 $y'' + \left(\frac{2}{x} - 2\right) y' = \left(\frac{2}{x} - 1\right) y$, $x \neq 0$

試問 $y(x) = ?$

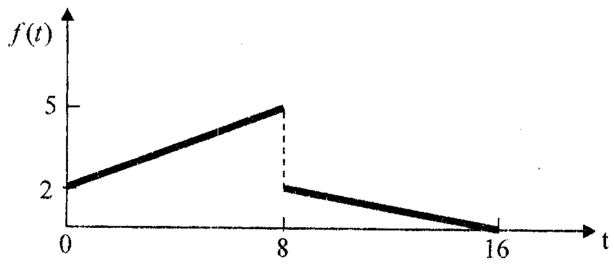
(須列出解題過程，否則不予計分)

所別：土木工程學系碩士班 甲組 科目：工程數學

1. (10%)

(a) (5%) 請用單位階梯函數(unit step function)表示下圖所示之函數 $f(t)$=?

(b) (5%) 請求下圖所示函數 $f(t)$ 之 Laplace 轉換 (transform)，$L(f(t))=F(s)$=?



2. (15%)

求解以下之四階常微分方程之通解 $y(x)=?$

$$y^{(4)}+10y^{(2)}+9y=2\sinh x$$

3. (15%)

向量 $\mathbf{A}=3y\vec{i}-xz\vec{j}+yz^2\vec{k}$，$S$ 爲以 $2z=x^2+y^2$ ($z\le 2$) 表示之曲面，C 爲曲面 S 之邊界曲綫 (boundary curve)，請求以下封閉積分

$$\oint_C \mathbf{A}\bullet d\mathbf{r}=?$$

(其中 $\mathbf{r}$ 爲邊界曲綫 C 上點之位置向量(position vector), $\vec{i},\vec{j},\vec{k}$ 分別爲 x, y, z 坐標軸上之單位向量)。

4. (20%)

設 A 與 B 皆爲 4 x 4 方陣，且 $B=(B_{ij})=A^n$ (n 爲正整數)。又設

$$A=(A_{ij})=\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}=\begin{pmatrix} -6 & 0 & 0 & 0 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & -3 \end{pmatrix}.$$

求出 B_{32} 的值。

5. (10%)

設 Γ 代表複數平面上的單位圓，其方程式爲 $|z|=1$，$z=x+iy$，$i=\sqrt{-1}$。請計算出閉迴路積分

$\oint e^{\frac{1}{z}}dz$。 提示： 可應用 $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ 公式去求出 $e^{\frac{1}{z}}$ 在 $z=0$ 位置上的留數(residue).

6. (10%)

(a)(3%) 一維熱傳導方程式屬於偏微分方程式的哪一類方程式？

(b)(4%) 傅氏轉換和傅氏級數用於求解一維熱傳導問題時，所解的問題有何不同？

(c)(3%) 在何種情形下，可以推得傅氏餘弦變換(Fourier Cosine Transform)？

7. (10%)

已知一條兩端固定的弦之長度爲 π，波速爲 1，若此弦受到初始位移 $f(x)=k\sin 3x$；初始速度 $g(x)=-0.01\sin x$，求其位移？

8. (10%)

利用 $f(x)=e^{-x}$ (x>0) 計算 $\int_0^{\infty}\frac{\cos x\omega}{1+\omega^2}d\omega$ $x>0$。

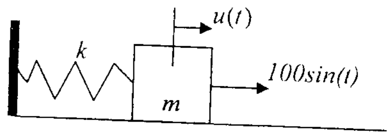
所別：土木工程學系碩士班 結構組　科目：工程數學　共 2 頁 第 1 頁

＊請在試卷答案卷（卡）內作答

參考用

1. (20%)

(a)(5%)　請寫出下圖所示之光滑面上的質點-彈簧系統之水平方向運動之控制方程式?



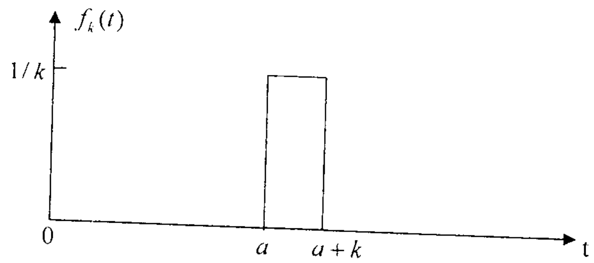
(b)(10%)　請求解(a)題系統在自由振動(free vibration)下，具有初始位移$u(0)=\Delta$，初始速度$\dot{u}(0)=\alpha$時之解$u(t)=?$

(c) (5%)　請問(b)題中初始時刻($t=0$)之相位角δ是多少

2. (15%)

(a) (5%)　請求下圖所示函數$f_k(t)$之 Laplace 轉換 (transform)，$L(f_k(t))=F_k(s)=?$

(b) (10%)　Limit　$F_k(s)=?$ as　$k\to 0$



3. (15%) 請求解以下之聯立常微分方程，$y_1(t)=?$，$y_2(t)=?$

$$\begin{Bmatrix}\ddot{y}_1(t)\\ \ddot{y}_2(t)\end{Bmatrix}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}\begin{Bmatrix}y_1(t)\\ y_2(t)\end{Bmatrix}$$

注意：背面有試題

參考用

4) 一圓柱体外表由一圓柱面及兩平面所組成。圓柱面方程為 $x^2+y^2=16$，兩平面的方程分別為 $z=1$ 和 $z=5$。設 $d\underset{\sim}{A}$ 代表此圓柱体表面的面積微元，$d\underset{\sim}{A}$ 為一向量。另外，$\underset{\sim}{v} = y^2\underset{\sim}{i} + xz^3\underset{\sim}{j} + (z-1)^2\underset{\sim}{k}$ 代表一向量場，請計算出表面積分 (surface integral) $\iint_S \underset{\sim}{v}\cdot d\underset{\sim}{A}$。

此積分的下標 s 代表圓柱体的表面。S 由圓柱面和兩平面所組成　(25%)

5) 設 $f(t) = \begin{cases} -\cos \pi t, & -1 < t < 0 \\ \cos \pi t, & 0 < t < 1 \end{cases}$。

而且 f(t) 為週期函數，其週期為 2。

請計算出此週期函數的 Fourier 級數的係數。　(25%)

注意：背面有試題

中央大學

機械工程學系

91~97 學年度
工程數學考古題

國立中央大學九十一學年度碩士班研究生入學試題卷

所別：機械工程學系甲、乙、丙、已組　科目：工程數學　共 1 頁　第 1 頁

1. (a) Solve the initial value problem

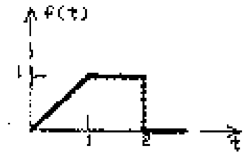
$$ay'' + by = 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

where a and b are constants, but $a \neq 0$. (12%)

(b) Find a basis of solution of the differential equation. (Show the details of your work.) (8%)

$$x^2 y'' + 3xy' + y = 0$$

(c) Find the Laplace transforms of the following function. (Show the details of your work.) (5%)



2. (a) Evaluate $\oint_c \frac{e^z}{(z-1)(z+4)}\, dz$, where c is the circle $|z| = 3$ described in the positive direction. (8%)

(b) Evaluate $\oint_c z^6 \sin(1/z)\ dz$, where c is the circle $|z| = 1$ described in the positive direction. (7%)

(c) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta$. (10%)

3. (a) Find the similarity transformation $A = P\Lambda P^{-1}$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and Λ is a diagonal matrix. (10%)

(b) Consider a system of differential equations $\frac{dy}{dx} = Ay$ subject to the initial condition $y(0) = b$, where the matrix A is given as above, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can solve this problem by taking the iterative procedure:

$$y^{(0)} = b,$$

$$y^{(1)} = b + \int_0^x Ay^{(0)} d\xi = b + xAb,$$

$$y^{(2)} = b + \int_0^x Ay^{(1)} d\xi = b + xAb + \frac{(xA)^2}{2!} b,$$

$$\vdots$$

$$y^{(n)} = \left[I + \frac{xA}{1!} + \frac{(xA)^2}{2!} + \cdots + \frac{(xA)^n}{n!} \right] b,$$

and $y^{(n)} \to y$ as $n \to \infty$. Obtain y_1 and y_2 by the iteration method and the similarity transformation you have got. Show the details of your work. (Hint: think about the Taylor series expansion for e^t about $t = 0$.) (15%)

4. By the method of separation of variables, find the solution $u(x, y)$ of the Poisson equation

$$u_{xx} + u_{yy} = \cos(\pi y),$$

in the semi-infinite strip $0 \le x < \infty, 0 \le y \le 1$, such that

$$u(0, y) = y, \quad u_y(x, 0) = u_y(x, 1) = 0. \qquad (25\%)$$

參考用

系所別：機械工程學系 甲組乙組丙組己組 科目：工程數學

參考用

1. Let

$$\mathbf{A}=\begin{bmatrix} 1 & 2+i & 3-2i & 4+3i & 5-4i \\ 2-i & 2 & 4-3i & 5+4i & 6-5i \\ 3+2i & 4+3i & 3 & 6+5i & 7-6i \\ 4-3i & 5-4i & 6-5i & 4 & 2 \\ 5+4i & 6+5i & 7+6i & 2 & 5 \end{bmatrix}_{5\times 5}$$

Prove that all eigenvalues of $\mathbf{A}$ are real. (6%)

2. For the linear system of equations $\mathbf{Ax}=\mathbf{b}$, where

$$\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & k_1 \\ 3 & k_2 & 0 \\ 4 & 5 & 10 \end{bmatrix},\quad \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\text{ and } \mathbf{b}=\begin{bmatrix} 1 \\ b_2 \\ 3 \\ 4 \end{bmatrix},$$

(a) determine the values of k_1, k_2, and b_2, for which the system has infinitely many solutions; (4%)

(b) determine the values of k_1, k_2, and b_2, for which the system has precisely one solution with $x_3 \neq 0$; (4%)

(c) determine the values of k_1, k_2, and b_2, for which the system has precisely one solution with $x_1 = 1$. (4%)

3. Let $\mathbf{D}=\mathbf{x}^{-1}\mathbf{A}\mathbf{x}$ be diagonal, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal.

(a) Prove that

$$\mathbf{D}^m=\mathbf{x}^{-1}\mathbf{A}^m\mathbf{x} \quad (m=2, 3, \ldots) \qquad (3\%)$$

(b) Find $\mathbf{A}^{10}$ where $\mathbf{A}=\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. (4%)

4.

(a) Solve the problem

$$\frac{dy}{dx}=\frac{-\sin y}{(1+x)\cos y} \qquad (4\%)$$

(b) Solve the problem

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \qquad (5\%)$$

(c) Find the general solution to the equation

$$\frac{d^2y}{dx^2}+y=\cos x \qquad (6\%)$$

(d) What is the relationship between the Fourier transformation and the Laplace transformation? Can the function $f(x)=x3^x$ be transformed by the two methods? Explain. (10%)

注意：背面有試題

系所別: 機械工程學系 甲組乙組丙組己組　科目: 工程數學

5.

(a) Show that the vector field $\vec{F} = (x+2y)\vec{i} + (2x-y)\vec{j}$ is a gradient field. Find a potential function for $\vec{F}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, C: (1,0) to (3,2). (6%)

(b) Evaluate the line integral

$$\oint_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}, \text{ where } C \text{ is the ellipse } x^2+4y^2=4. \quad (9\%)$$

(c) Use Stokes's theorem to evaluate

$$\oint_C z^2e^{x^2}dx + xy^2dy + \tan^{-1} ydz,$$

where C is the circle $x^2+y^2=9$, by finding a surface S with C as its boundary and such that the orientation of C is counterclockwise. (10%)

6.

(a) Use separation of variables to find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial u}{\partial y}$. (10%)

(b) Use the Laplace transform to solve the boundary-value problem (15%)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\quad 0<x<1,\quad t>0,$$

$$u(0,t)=0,\quad u(1,t)=0,$$

$$u(x,0)=0,\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 2\sin\pi x + 4\sin 3\pi x.$$

所別：機械工程學系碩士班 甲、乙組 丙組 科目：工程數學

Ordinary Differential Equation (33%)

1. (a) Please use Laplace Transform to solve the ordinary differential equation of $y''+25y=5\delta(t-\pi)$, with initial conditions of $y(0)=3$ and $y'(0)=0$. Note that δ is the Dirac delta function. (12%)

   (b) Calculate the values of $y(\pi/2)$ and $y(2\pi)$. (3%)

2. Solve $y'+y=-2x/y$ with initial condition of $y(0)=2$. (10%)

3. Solve $y''-9y=0$ with $y(0)=1$ and $y'(0)=0$. Please present your answer in the form of Hyperbolic function. (8%)

Linear Algebra & Vector Calculus (33%)

4. For the linear system $\mathbf{Ax}=\mathbf{b}$, where the matrix $\mathbf{A}=[a_{ij}]_{3\times4}$ is given by

$$\mathbf{A}=\begin{bmatrix}-1&5&-1&-3\\4&-1&2&6\\3&4&1&3\end{bmatrix}\quad \mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\quad\text{and}\quad \mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

   (a) Find all the possible vectors $\mathbf{b}$ for which the linear system has non-trival solution. (5%)

   (b) Determine the solution $\mathbf{x}$. (5%)

參考用

5. Use Green'theorem to evaluate $\oint_C \vec{F}\cdot d\vec{R}$, where $\vec{F}=xy\vec{i}+xy^2\vec{j}$ and C: the triangle with vertices $(0,0)$, $(3,0)$, $(0,5)$. Note that the curve C is oriented counterclockwise. (8%)

6. Determine the surface area of a sphere of radius a using the technique of surface integral. (8%)

7. Let $\{v_1,v_2\}$ span the vector space of inner product in R^2. Please answer the following questions.

   (a) Is it true that v_1 and v_2 must be mutually orthorgonal? Explain why or why not. (3%)

   (b) Give two examples showing that $\{v_1,v_2\}$ is a orthornormal base in R^2. (4%)

注意：背面有試題

所別：機械工程學系碩士班 甲、乙組 丙組 科目：工程數學

Fourier Analysis, Partial Differential Equation and Complex Analysis (34%)

8. The function

$$f(x)=\begin{cases}-1, & -\pi<x<0\\ 1, & 0\le x<\pi\end{cases}$$

(a) Expand $f(x)$ in a Fourier series. (5%)

(b) Expand $f(x)$ in a complex Fourier series. (5%)

9. (a) Solve the partial differential equation (4%)

$$\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=0$$

(b) Solve the boundary-value problem (10%)

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\qquad 0<x<\pi,\qquad t>0$$

$$u(0,t)=0,\qquad u(\pi,t)=0,\qquad t>0$$

$$u(x,0)=\sin x,\qquad 0<x<\pi.$$

10. Using residue calculus, evaluate (10%)

$$I=\int_0^{2\pi}\frac{\cos 2\theta}{5-4\sin\theta}d\theta.$$

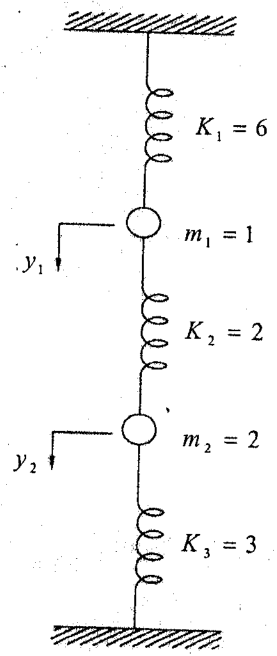
參考用

(K)

所別：法國語文學系碩士班 不分組 科目：法文議論文

Ordinary Differential Equation (33 %)

1. Consider the mass-spring system as shown in the figure. Assume that there is no damping and that no external force is applied to the system. Suppose that the upper weight is pulled down two units and the lower weight is raised one unit, then both weights are released from rest simultaneously at time $t = 0$.
   (1) Please derive the system of two second order differential equations governing the position of the weights relative to their equilibrium positions at any time $t = 0$. Note that only the system of differential equation is required. (5%)
   (2) Please convert the system of two second order differential equations, you have obtained in (1), into a system of four first order differential equations. (5%)
   (3) Let the system of four first order differential equations be written as $X' = AX$. Determine A and $X(0)$? (5%)
   (4) Determine the eigenvalues of A. (5%)



2. Suppose that the differential equation $P(x,y)dx + Q(x,y)dy = 0$ is not exact.
   (1) What is the necessary condition for the differential equation to have an integrating factor $F = F(x)$. (4%)
   (2) Let $P(x,y) = -y$ and $Q(x,y) = x$. Determine an integrating factor $F = F(y)$ of the differential equation. (4%)
3. Given

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu}\, m!\,\Gamma(\nu + m + 1)}$$

where $J_\nu(x)$ is known to be the Bessel function of the first kind of order ν, and Γ the Gamma function. Show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \qquad (5\%)$$

Note that you may directly use $\Gamma(1/2) = \sqrt{\pi}$ without proof.

注意：背面有試題

Linear Algebra & Vector Calculus (33 %)

4. Using Green's theorem, evaluate the line integral $\oint_C \mathbf{F}(\mathbf{r})d\mathbf{r}$ counterclockwise around the boundary C of a closed region R, where
$\mathbf{F} = [x\cosh(2x), x^2\sinh(2y)]$, R: $x^2 \le y \le x$. (10%)

5. Evaluate surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}dA$ for the following data:
$\mathbf{F} = [6x^2, 4y^2, 0]$ S: $\mathbf{r} = [u, v, 3u+6v]$, $1 \le u \le 2$, $-2 \le v \le 2$. (15%)

6. Let $\mathbf{F} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, find a symmetric matrix $\mathbf{B}$ and a skew-symmetric matrix $\mathbf{C}$, such that $\mathbf{B} + \mathbf{C} = \mathbf{A}$. (8%)

Fourier Analysis, Partial Differential Equation and Complex Analysis (34 %)

7. (a) Expand $f(x) = x + \pi$, $-\pi < x < \pi$ in a Fourier series. (7%)

(b) Use the result of (a) to find $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ (3%)

8. (a) Solve the wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \; 0 < x < L, \; t > 0$$

$$u(0, t) = 0, \; u(L, t) = 0, \; t > 0$$

$$u(x, 0) = f(x), \; \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \; 0 < x < L \; (5\%)$$

(b) Show that the solution of (a) can be written as $u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)]$. (5%)

9. (a) Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series that is valid in a deleted neighborhood of $z = 1$. State the domain throughout which the series is valid. (3%)

(b) Find $\oint_C \dfrac{1}{z(z-1)}dz$, where C is the circle $|z-1| = 6$, by means of the residue theorem. (3%)

(c) Find $\int_0^\infty \dfrac{dx}{x^6+1}$. (8%)

所別：機械工程學系碩士班 甲組(固力與設計) 科目：工程數學
乙組(製造與材料)
丙組(熱流)
能源工程研究所碩士班

Ordinary Differential Equation (33 %)

1. Assuming that the radium decomposes at a rate proportional to the amount present, in how many years will half the original amount be lost if 10% disappears in 263 years. (8%)

2. (a) Show that $y_1 = x$ and $y_2 = x^2$ are both linearly independent solutions of $x^2y'' - 2xy' + 2y = 0$ (4%)

   (b) Find the particular solution for which $y(1) = 3$ and $y'(0) = 5$. (4%)

3. Compute by direct evaluation of the integral the Laplace transform $L[f(x)]$

   (a) $f(x) = \sin 3x$ (4%)

   (b) $f(x) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t > 1 \end{cases}$ (4%)

4. Show that for the hypergeometric series $F(a,b,c,x)$,

   $$\frac{dF(a,b,c,x)}{dx} = \frac{ab}{c}F(a+1,b+1,c+1,x) \qquad (9\%)$$

   (Hint: the hypergeometric series can be expressed as follows

   $$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}x^n)$$

Linear Algebra & Vector Calculus (33 %)

5. Find the surface integrate $\iint_S \mathbf{F}\cdot\mathbf{n}dA$, when $\mathbf{F} = [x^2,\ 0,\ 2y^2\ ]$，S is the portion of the plane $3x+2y+z=6$ in the first octant $(x \ge 0, y \ge 0, z \ge 0)$. (15%)

6. If $\mathbf{F} = \frac{2x}{x^2+y^2}\mathbf{i} + \frac{2y}{x^2+y^2}\mathbf{j}$, $C_1 : (x-2)^2 + (y-1)^2 = 1$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, find the integral $\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}$ counterclockwise around C_1. (8%)

7. Consider the matrix **A**, Determine matrices **Q** and **D** such that **Q**$^{-1}$**AQ=D** is diagonal. (10%)

   $$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

注意：背面有試題

所別：機械工程學系碩士班 甲組(固力與設計) 科目：工程數學
乙組(製造與材料)
丙組(熱流)
能源工程研究所碩士班

Fourier Analysis, Partial Differential Equation and Complex Analysis (34 %)

8. Find the Fourier series of

$$f(x)=\begin{cases}0 & -2<x<0\\ x & 0\le x<1\\ 1 & 1\le x<2\end{cases}\qquad (10\%)$$

9. (a) Use separation of variables to find the product solutions of

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad k>0 \qquad (4\%)$$

(b) Find the temperature $u(x,t)$ in a rod of length 2 if the initial temperature is

$f(x)=\begin{cases}x & 0<x<1\\ 0 & 1<x<2\end{cases}$ throughout and if the ends $x=0$ and $x=2$ are insulated. (10%)

10. (a) Use a Laurent series to find the indicated residue.

$$f(z)=\frac{e^{-z}}{(z-2)^2}\ ;\ \mathrm{Res}(f(z),2) \qquad (5\%)$$

(b) Evaluate $\int_{-\infty}^{\infty}\frac{\cos 2x}{x^2+1}dx$ (5%)

所別：機械工程學系碩士班 甲、乙、丙組　　科目：工程數學　共 2 頁 第 1 頁
能源工程研究所碩士班
光機電工程研究所碩士班 乙組(光機組)
生物醫學工程研究所碩士班
*請在試卷答案卷（卡）內作答

參考用

Ordinary Differential Equation (25 %)

1. Find the family of the curve such that the projection on the x-axis of the part of the tangent between (x, y) and the x axis has length 1. (5%)

2. A 6 *lb.* weight is attached to the lower end of a spring suspended from the ceiling, the spring constant being $27\,lb/ft$. The weight comes to rest, and beginning at $t = 0$ an external force given by $F(t) = 12\cos 20t$ is applied to the system. Determine the resulting displacement as a function of time, assuming damping is negligible. (10%)

3. How many methods can you use to solve the differential equation

$$2xydx + (y^2 - x^2)dy = 0$$

Explain your answers. (10%)

Linear Algebra & Vector Calculus (25 %)

4. Show that the differential form under the integral sign of

$$I = \int_{(-1,5)}^{(4,3)} (3z^2 dx + 6xzdz)$$

is exact, so that we have independence of path in any domain, and find the value of the integral I from A: (-1, 5) to B: (4, 3). (10%)

5. Find out what type of conic section is represented by the given quadratic form.
$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$. Transform it to principal axes. (15%)

注意：背面有試題

Complex Analysis (25 %)

6. Determine where the function, $f(z) = 2x - x^3 - xy^2 + i(x^2 + y^3 - 2y)$, is analytic. (10%)

7. Evaluate the following integral counterclockwise. (15%)

$$\oint_C \cot\frac{z}{4}\,dz, \qquad C: |z| = 1.$$

Partial Differential Equation and Fourier Analysis (25 %)

8. Show that the Fourier series of $f(x) = x$, $-\pi < x < \pi$ leads to

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \qquad (10\%)$$

9. Solve the partial differential equation (15%)

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - 1, \quad 0 < x < 1, \quad t > 0$$

$$T(x,0) = \frac{x^2}{2} + \cos(\pi x), \quad 0 < x < 1$$

$$\frac{\partial T(0,t)}{\partial x} = 0, \quad \frac{\partial T(1,t)}{\partial x} = 1, \quad t > 0.$$

注意：背面有試題